2. Irrational and Real numbers

Let $\mathbb{Q}^c$ be a complement set of $\mathbb{Q}$ in real number $\mathcal{R}$. $\mathbb{Q}^c = \mathcal{R}\backslash\mathbb{Q} = \{x \in \mathcal{R}: x \notin \mathbb{Q}\}$, $\mathbb{Q}^c$ is called the set of irrational numbers, since $\mathbb{Q}^c \neq \mathbb{Q} \Longrightarrow \sqrt{2} \in \mathbb{Q}^c$.

(2.1)**Theorem:** Let $x \in \mathbb{Q}$ and $y \in \mathbb{Q}^c$, then

(1) $x + y \in \mathbb{Q}^c$.
(2) $xy \in \mathbb{Q}^c$, with $x \neq 0$.

Proof:(1) Assume that $x + y \notin \mathbb{Q}^c$, since $x + y \in \mathcal{R} \Longrightarrow x + y \in \mathbb{Q}$, since $x \in \mathbb{Q}$ and $\mathbb{Q}$ is a field $\Longrightarrow -x \in \mathbb{Q}$, also $(x + y) + (-x) \in \mathbb{Q} \Longrightarrow y \in \mathbb{Q}$, but this is a contradiction.

2)Let $xy \notin \mathbb{Q}^c \Longrightarrow xy \in \mathbb{Q}$, since $\mathbb{Q}$ is a field and $x \in \mathbb{Q}$, $x \neq 0 \Longrightarrow \frac{1}{x} = x^{-1} \in \mathbb{Q}$, also $\frac{1}{x}(xy) \in \mathbb{Q} \Longrightarrow y \in \mathbb{Q}$, but this is a contradiction. ■

(2.2)**Theorem:**(Density of irrational numbers)

Let $a, b \in \mathcal{R} \ni a < b \ \exists\, s \in \mathbb{Q}^c \ni a < s < b \Longrightarrow \exists$ an infinity irrational numbers between any two real numbers.

Proof: Since $a < b \Longrightarrow a - \sqrt{2} < b - \sqrt{2}$, since $a - \sqrt{2}$ and $b - \sqrt{2}$ are real numbers $\Longrightarrow$ by using density of rational numbers $\Longrightarrow \exists r \in \mathbb{Q} \ni a - \sqrt{2} < r < b - \sqrt{2} \Longrightarrow a < r + \sqrt{2} < b \Longrightarrow$ since $r \in \mathbb{Q}$ and $\sqrt{2} \in \mathbb{Q}^c \Longrightarrow s = r + \sqrt{2} \in \mathbb{Q}^c \Longrightarrow a < s < b$. Now, since $a < s \Longrightarrow \exists\, s_1 \in \mathbb{Q} \ni a < s_1 < s$, by continuing this operation, we get on an infinite number of irrational numbers located between a, b. ■

(2.3) **Definition**: Let $a, b \in \mathcal{R} \ni a < b$, then

$(a, b) = \{x \in \mathcal{R}: a < x < b\}$

$[a, b] = \{x \in \mathcal{R}: a \leq x \leq b\}$

$(a, b] = \{x \in \mathcal{R}: a < x \leq b\}$

$[a, b) = \{x \in \mathcal{R}: a \leq x < b\}$

$(-\infty, b) = \{x \in \mathcal{R}: -\infty < x < b\}$

$(-\infty, b] = \{x \in \mathcal{R}: -\infty < x \leq b\}$

$(a, \infty) = \{x \in \mathcal{R}: a < x \leq \infty\}$

$[a, \infty) = \{x \in \mathcal{R}: a \leq x < \infty\}$.

(2.4) **Note:** According to density of rational and irrational numbers, we can say that every interval of real numbers contains an infinite number of rational and irrational.

(2.5) **Definition**: (**Absolute Value**) Let x be a real number, absolute value of x is denoted by $|x|$ and defined as:

$$|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

(2.6)**Theorem:**(Properties of Absolute Value)

1. $|x| = \max\{-x, x\}\ \forall x \in \mathcal{R} \Longrightarrow |x| \geq -x, |x| \geq x$.
2. $|x| \geq 0\ \forall\ x \in \mathcal{R}$.
3. $|x| = 0$ iff $x = 0$.
4. $|x| = |-x| \forall\ x \in \mathcal{R}$.
5. $|x - y| = |y - x|\ \forall\ x, y \in \mathcal{R}$.
6. $|xy| = |x||y|\ \forall\ x, y \in \mathcal{R}$.
7. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}\ \ \forall\ x, y \in \mathcal{R}, y \neq 0$.
8. $|x + y| \leq |x| + |y|\ \forall\ x, y \in \mathcal{R}$.
9. $|x - y| \leq |x| + |y|\ \forall\ x, y \in \mathcal{R}$.
10. $\big||x| - |y|\big| \leq |x - y| \forall\ x, y \in \mathcal{R}$.
11. $|x| \leq a$ iff $-a \leq x \leq a$.

Some Important Inequalities

(2.7)**Theorem**:

1. **Cauchy-schwars Inequality.**

If $p, q \in \mathcal{R} \ni \frac{1}{p} + \frac{1}{q} = 1 \Longrightarrow \sum_{i=1}|x_i y_i| \leq (\sum_{i=1}|x_i|^p)^{\frac{1}{p}} (\sum_{i=1}|y_i|^p)^{\frac{1}{p}} \ni x_i, y_i \in \mathcal{R}$. In particular, if $p = 2 \Longrightarrow q = 2$ and $\sum_{i=1}|x_i y_i| \leq (\sum_{i=1}|x_i|^2)^{\frac{1}{2}} (\sum_{i=1}|y_i|^2)^{\frac{1}{2}}$.

2. **Minkokowsks Inequality.**

If $p \geq 1 \Longrightarrow (\sum_{i=1}|x_i + y_i|^p)^{1/p} \leq (\sum_{i=1}|x_i|^p)^{\frac{1}{p}} + (\sum_{i=1}|y_i|^p)^{\frac{1}{p}} \ni x_i, y_i \in \mathcal{R}$.

Countable Sets.

(2.8) **Definition**: Let A, B be a sets. We say that A is an equivalent to B and written by $A\sim B$, if there is a bijective function from A into B and written $A \nsim B$, if A is an inequivalent to B.

(2.9)**Theorem**:

1. If $E = \{2,4,6, \dots\}$, then $\mathbb{N}\sim E$.

($f: \mathbb{N} \longrightarrow E$ defined by $f(n) = 2n\ \forall\ n \in \mathbb{N}$)

2. If $O = \{1,3,5, \dots\}$, then $\mathbb{N}\sim O$.

($f: \mathbb{N} \longrightarrow O$ defined by $f(n) = 2n - 1\ \forall\ n \in \mathbb{N}$)

3. If $\mathbb{N}^* = \{0,1,2,3, \dots\}$, then $\mathbb{N}\sim\mathbb{N}^*$.

($f: \mathbb{N} \longrightarrow \mathbb{N}^*$ defined by $f(n) = n - 1\ \forall\ n \in \mathbb{N}$)

4. $\mathbb{Z}\sim\mathbb{N}^*$. ($f: \mathbb{Z} \longrightarrow \mathbb{N}^*$ defined by $f(x) = \begin{cases} -2x & , x \leq 0 \\ 2x - 1, & x > 0 \end{cases}$)
5. $\mathbb{N}\sim\mathbb{Q}$.

We deduce that $E, O, \mathbb{N}, \mathbb{N}^*, \mathbb{Z}, \mathbb{Q}$ are equivalent.

6. If $A = [0,1]$, $I_1 = (a, b), I_2 = (a, b], I_3 = [a, b), I_4 = [a, b]$ then $A\sim I_i \forall i = 1,2,3,4$.

($f: A \longrightarrow I_i$ defined by $f(x) = a + (b - a)x\ \forall\ x \in A$)

7. If $A = (-1,1)$ and $B = (a, b)$, then $A\sim B$.

($f: A \longrightarrow B$ defined by $f(x) = \frac{1}{2}(b - a)x + \frac{1}{2}(b + a)\ \forall\ x \in A$)

8. If $A = (0,1)$ then $A\sim\mathcal{R}^+$.

($f: A \longrightarrow \mathcal{R}^+$ defined by $f(x) = \frac{x}{1-x} \forall\ x \in A$)

9. If $A = (-1,1)$ then $A\sim\mathcal{R}$.

($f: A \longrightarrow \mathcal{R}$ defined by $f(x) = \sin x\ \forall\ x \in A$)

10. If $A = (\frac{-\pi}{2}, \frac{\pi}{2})$ then $A\sim\mathcal{R}$.

($f: A \longrightarrow \mathcal{R}$ defined by $f(x) = \tan x\ \forall\ x \in A$)

11. If $A = (0,1)$ then $A \sim \mathcal{R}$.
12. For all $k \in \mathbb{N}$ put $\mathbb{N}_k = \{1,2,3,\dots,k\}$ then
    a. $\mathbb{N}_k \nsim \mathbb{N}$.
    b. $\mathbb{N}_k \sim \mathbb{N}_1$ iff $k = 1$.
13. $P(X) \nsim X$ $\forall$ set X.

(2.10) **Definition**: Let A be a set. We say that A is a finite set, if A is a non-empty set or equivalent to $\mathbb{N}_k$ for some $k \in \mathbb{N}$. We say that A is an infinite set, if A does not finite set.

(2.11) **Definition**: If A is a finite set, then $A \sim \mathbb{N}_k$ for some $k \in \mathbb{N}$ and then there is a bijective function $f: \mathbb{N}_k \longrightarrow A$, put $f(i) = a_i \ \forall i \in \mathbb{N}_k \Longrightarrow a_i \in A \ \forall i = 1,2,3,\dots,k$ and then $A = \{a_1, a_2, \dots, a_k\}$.

(2.12)**Theorem**:

1. Let A, B be a non-empty sets such that $A \sim B$ then
   a. A is a finite iff B is a finite.
   b. A is an infinite iff B is an infinite.
2. For all finite set inequivalent to proper subset.
3. Every subset of finite set be a finite.
4. If A is an infinite set and $A \subset B$ then B is an infinite set.
5. If A is an infinite set and B is a set then $A \cup B$ is an infinite.

(2.13) **Definition**: Let A is a set. We say that A be a countable set, if A be a finite or equivalent to $\mathbb{N}$. We say that A be an infinite and countable, if A be an infinite and equivalent to $\mathbb{N}$. We say that A be an uncountable, if A be an infinite and inequivalent to $\mathbb{N}$.

(2.14)**Theorem**:

1. Every finite set is a countable.
2. Each of $O, E, \mathbb{N}, \mathbb{N}^*, \mathbb{Z}, \mathbb{Q}$ be an infinite and countable set.
3. Each of $\mathcal{R}$ and an intervals of $\mathcal{R}$ are an uncountable sets.

(2.15) **Note**: If A be an infinite countable set, then $\mathbb{N} \sim A$ and then there is a bijective function $f: \mathbb{N} \longrightarrow A$, put $f(n) = a_n \ \forall n \in \mathbb{N}$ and then $A = \{a_n : n \in \mathbb{N}\} = \{a_1, a_2, a_3, \dots\}$.

(2.16)**Theorem**:

1. Every countable infinite set be an equivalent to a proper subset.
2. Every infinite set contains a countable infinite subset.
3. The set $\mathbb{N} \times \mathbb{N}$ be a countable.
4. If A, B are a countable sets, then

a. $A \cup B$ be a countable.

b. $A \times B$ be a countable.