Lecture (5)

Explicit Method of Finite Difference

5.1 Introduction

In physics and mathematics, heat equation is a special case of diffusion equation and is a partial differential equation (PDE). A very popular numerical method known as finite difference methods (explicit and implicit schemes) is applied extensively for solving heat equations successfully. Examples of Explicit schemes are *Forward Time and Centre Space (FTCS)*, *Dufort* and *Frankel methods*, whereas examples of implicit schemes are *Laasonen* and *Crank-Nicolson* methods.

5.2 An Explicit Method of Solution

The explicit method of solution describe an unknown value at a certain grid point depending on the known values at neighboring grid points. By using equations (4.11) and (4.14) in the diffusion equation (which is of parabolic type), we get:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \quad ,$$

and this can be written as:

$$u_{i,j+1} = u_{i,j} + r\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) \qquad (5.1)$$

where $r = \frac{k}{h^2}$

Hence, we can calculate u-values at the first time row depending on the boundary and initial values at t=0 line. Then, similarly we can calculate the values at the second time row depending on the values that was calculated for the first row, and so on. This is called the explicit method.

Example: Consider a metal rod is heated for a long time in the center and keeping the ends in contact with blocks of melting ice and that the initial temperature distribution in non-dimensional form (unitless) is:

$$(a) \quad u = 2x, \qquad 0 \le x \le \frac{1}{2},$$

$$(b) \quad u = 2(1 - x), \quad \frac{1}{2} \le x \le 1.$$

In other words we are seeking a numerical solution of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ which satisfies,

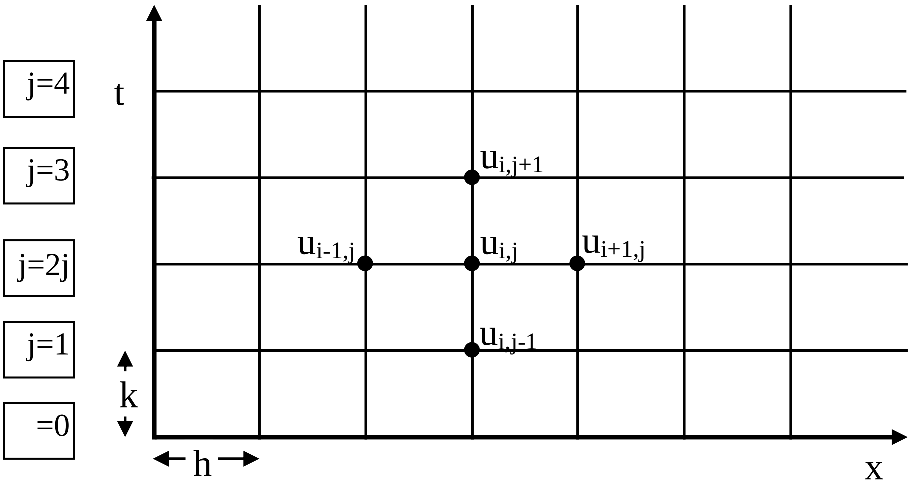
1. $u = 0 \;\; when\; x = 0 \; and \; 1 \quad for\; all\; t. \quad (the\; boundary\; conditions)$
2. $u = 2x\; for \;\; 0 \le x \le \frac{1}{2}, \quad and \quad u = 2(1-x) \;\; for \; \frac{1}{2} \le x \le 1$

$at \;\; t = 0 \quad (\, the\; initial\; conditions)$

For h=1/10 , the initial values and boundary values are as shown in in table (5.1). The problem is symmetric with respect to x=1/2 , so we need the solution only for $0 \leq x \leq \frac{1}{2}$.

Table 5.1

	X=0	0.1	0.2	0.3	0.4	0.5	0.6 x
j=0	0	0.2	0.4	0.6	0.8	1.0	0.8
j=1	0						
j=2	0						
j=3	0						
j=4	0						



Take h=1/10, k=1/1000 so $r = \frac{k}{h^2} = \frac{1}{10}$, equation (5.1) which is

$$u_{i,j+1} = u_{i,j} + r\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

becomes:

$$u_{i,j+1} = \frac{1}{10}\left(u_{i-1,j} - 8u_{i,j} + u_{i+1,j}\right) \qquad (5.2)$$

For hand calculation, the "molecule" in Fig (5.2) can represent the relationship among these four functional values

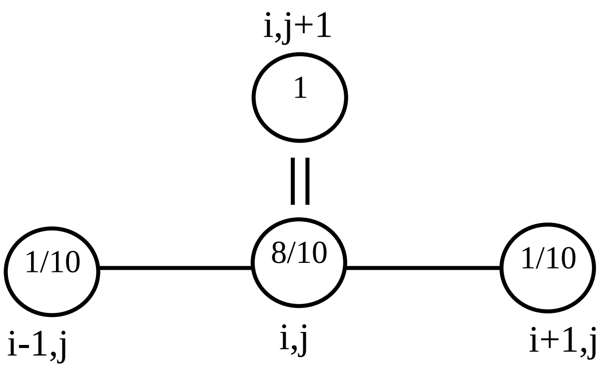


Fig. (5.2) Molecular representation

The numbers in the circles (atoms) are the multiplicands the functional values at the corresponding grid points.

The application of equation (5.2) on the data in Table (5.1) is shown in Table (5.2). For example:

$u_{5,1}$=1/10{0.8+(8×1)+0.8}=0.9600

$u_{4,2}$=1/10{0.6+(8×0.8)+0.96}=0.7960

Table 5.2

i=0 x = 0		i= 1 0.1	i= 2 0.2	i= 3 0.3	i =4 0.4	i= 5 0.5	i= 6 0.6
(j = 0) t=0.000	0	0.2000	0.4000	0.6000	0.8000	1.0000	0.8000
(j = 1) t=0.001	0	0.2000	0.4000	0.6000	0.8000	**0.9600**	0.8000
(j = 2) t=0.002	0	0.2000	0.4000	0.6000	**0.7960**	0.9280	0.7960
(j = 3) t=0.003	0	0.2000	0.4000	0.5996	0.7896	0.9016	0.7896
(j = 4) t=0.004	0	0.2000	0.4000	0.5986	0.7818	0.8792	0.7818
(j = 5) t=0.005	0	0.2000	0.3999	0.5971	0.7732	0.8597	0.7732
⋮							
(j =10) t=0.010	0	0.1996	0.3968	0.5822	0.7281	0.7867	0.7281
⋮							
(j=20) t=0.020	0	0.1938	0.3781	0.5373	0.6486	0.6891	0.6486

The analytical solution of the partial differential equation which fulfill the boundary and initial conditions is:

$$u = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{1}{2}n\pi\right)(\sin n\pi x)\exp(-n^2\pi^2 t)$$

The comparison of this solution with the finite difference solution at x= 0.3, shows that the finite difference solution of acceptable accuracy as in Table (5.3).

Table 5.3

Time step	Finite – difference Solution (x=0.3)	Analytical solution (x=0.3)	Difference	Percentage error
t=0.005	0.5971	0.5966	0.0005	0.08
t=0.01	0.5822	0.5799	0.0023	0.4
t=0.02	0.5373	0.5334	0.0039	0.7
t=0.10	0.2472	0.2444	0.0028	1.1

5.2 Exercises

Q1. For the following PDE: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, find the numerical solution at the grid points in the first time row by using the explicit method. Consider the following boundary conditions: u=4.56 at x=0 and u=1.56 at x=1 at $t \geq 0$

u= 3(1.52 – x) for $0 \leq x \leq 1$ at t=0

knowing that h=2/10 , k=3/1000.

Then, suppose that u=0.668 (Numerical solution) and u=0.665 (Analytical solution) calculate the difference and percentage error.

Q2. Consider the differential equation, $\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}$ with the following boundary conditions: u=3x+1 for $0 \leq x \leq 1.2$ find the numerical solution at the grid points in the first row (Take $\Delta x = h = 3/10, \Delta t = k = 1/1000$)