Lecture (4)

Finite Difference Methods (Part 2)

4.1 Two dimensions finite difference method

So far, we have discussed one-dimensional finite differences, which implicate ordinary differential equations. However, most atmospheric sciences problems require using two-dimensional formulae, which necessarily contain partial differential equations (PDE's). The Laplacian operator is an example and which is existing in the diffusion equation:

$$\frac{\partial f}{\partial t} = K\nabla^2 f \qquad (4.1)$$

The two-dimensional Taylor series is:

$$f(x_{i+1}, y_{j+1}) \equiv f_{i+1,j+1}$$

$$= f_{i,j} + \left[(\Delta x)_{i,j}\frac{\partial}{\partial x} + (\Delta y)_{i,j}\frac{\partial}{\partial y}\right] f + \frac{1}{2!}\left[(\Delta x)_{i,j}\frac{\partial}{\partial x} + (\Delta y)_{i,j}\frac{\partial}{\partial y}\right]^2 f$$

$$+ \frac{1}{3!}\left[(\Delta x)_{i,j}\frac{\partial}{\partial x} + (\Delta y)_{i,j}\frac{\partial}{\partial y}\right]^3 f + \frac{1}{4!}\left[(\Delta x)_{i,j}\frac{\partial}{\partial x} + (\Delta y)_{i,j}\frac{\partial}{\partial y}\right]^4 f$$

$$+ \cdots \qquad (4.2)$$

which can be written out in a different form as:

$$f_{i+1,j+1} = f_{i,j} + \left[(\Delta x)_{i,j}\, f_x + (\Delta y)_{i,j}\, f_y\right]$$

$$+ \frac{1}{2!}\left[(\Delta x)^2{}_{i,j}\, f_{xx} + 2(\Delta x)_{i,j}(\Delta y)_{i,j}\, f_{xy} + (\Delta y)^2{}_{i,j}\, f_{yy}\right]$$

$$+ \frac{1}{3!}\left[(\Delta x)^3{}_{i,j}\, f_{xxx} + 3(\Delta x)^2{}_{i,j}(\Delta y)_{i,j}\, f_{xxy} + 3(\Delta x)_{i,j}(\Delta y)^2{}_{i,j}\, f_{xyy}\right.$$

$$\left. + (\Delta x)^3{}_{i,j}\, f_{yyy}\right] + \cdots \qquad (4.3)$$

Here we use the notation:

$$(\Delta x)_{i,j} \equiv x_{i+1,j} - x_{i,j} \quad and \quad (\Delta y)_{i,j} \equiv y_{i,j+1} - y_{i,j} \qquad (4.4)$$

and it is understood that all of the derivatives are evaluated at the point (x_i , y_j).

4.2 Second Order Differential Equations

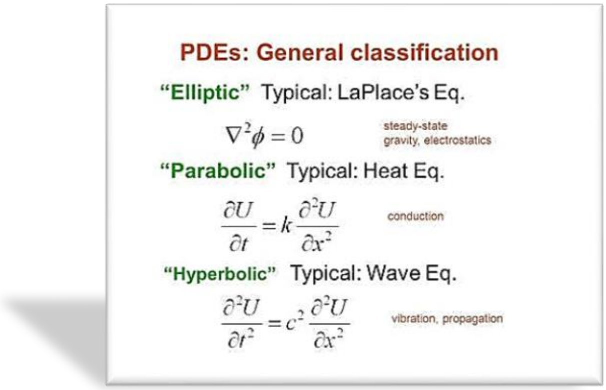
The mathematical formulation for most atmospheric sciences problems implies change rates with respect to two or more independent variables, mostly time and space. This will lead to partial differential equation or set of these equations. Let us take the following general second order equation with two dimensions x and y:

$$a\frac{\partial^2 \emptyset}{\partial x^2}+b\frac{\partial^2 \emptyset}{\partial x \partial y}+c\frac{\partial^2 \emptyset}{\partial y^2}+d\frac{\partial \emptyset}{\partial x}+e\frac{\partial \emptyset}{\partial y}+f\emptyset+g=0 \qquad (4.5)$$

Where a, b, c, d, e, f and g are either constants or more generally functions of x and y and of the dependent variable, $\emptyset$.

This equation is,

1. Elliptic when $b^2-4ac<0$
2. Parabolic when $b^2-4ac=0$
3. Hyperbolic when $b^2-4ac>0$



4.3 Three Important Equations

From equation (4.5):

1. If we put $\emptyset$=T, x=x, y=t, d=k , e=1 ,and the other coefficients a=b=c=f=g=0 we have the *advection equation*:

$$\frac{\partial T}{\partial t}+k\frac{\partial T}{\partial x}=0 \qquad (4.6)$$

Advection: is the transfer of heat or matter by the flow of a fluid, especially horizontally in the atmosphere or the sea.

2. If we put $\emptyset$=u, x=x, y=t, a=$c_{\circ}^2$, c=-1 , and the other coefficients b=d=e=f=g=0 we have the *wave equation*:

$$\frac{\partial^2 u}{\partial t^2}=c_{\circ}^2\frac{\partial^2 u}{\partial x^2} \qquad (4.7)$$

3. If we put $\emptyset$=T, x=x, y=t, a=k, e=-1 , and the other coefficients b=c=d= f=g=0 we have the *diffusion equation*:

$$\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2} \qquad (4.8)$$

Diffusion: is the net movement of anything from a region of higher concentration to a region of lower concentration. Diffusion is driven by a gradient in concentration.

The above three equations have great importance in science and can be classified into elliptic, parabolic, and hyperbolic (how?).

Determination of the equation type is essential because of its relation to the nature of the initial and boundary conditions in one hand, and how each PDE can be solved on the other hand.

4.3 Representing the dependent variables on the grid

Representation of variables on the rectangular mesh (or grid) is useful to understand and employ the finite difference method. The equations 4.6-4.8 are PDE's and contain two independent variables, ***x*** and ***t***.

To make a two dimensional grid, do:

1. Divide the plane of x and t into sets of rectangles.
2. Assume $\Delta x = h,\ \ \Delta t = k.$
3. Hence $x = ih \quad , t = jk \quad where\ i = 0,1,2,\ldots \quad j = 0,1,2,\ldots$

   If the dependent variable is u we can write the following Taylor expansions:

$$u_{i+1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} h + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} h^2 + \frac{1}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} h^3 + \cdots \qquad (4.9)$$

and

$$u_{i-1,j} = u_{i,j} - \left(\frac{\partial u}{\partial x}\right)_{i,j} h + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} h^2 - \frac{1}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} h^3 + \cdots \qquad (4.10)$$

It is clear that the two equations applies on x-axis. Now we can get the following equations from eq 4.9 and eq 4.10:

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \approx \frac{u_{i+1,j} - u_{i,j}}{h} \qquad forward\ \ difference \qquad (4.11)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \approx \frac{u_{i,j} - u_{i-1,j}}{h} \qquad backward\ \ difference \qquad (4.12)$$

Note that the 3rd and 4th terms on the right hand side were *truncated* with a truncation error of order O(h^2). That means the terms that contain the second order derivative and above were truncated.

Now, by subtracting eq 4.10 from 4.9 and neglecting the terms that contain 3rd derivative and above we can get:

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h} \qquad centered\ difference \qquad (4.13)$$

Adding the equations 4.9 to 4.10 yields a formula of second order derivative:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \qquad (4.14)$$

Note that the same equations can be written to the other coordinate (t).
The finite difference grid can be graphically represented as in figure (4.1).

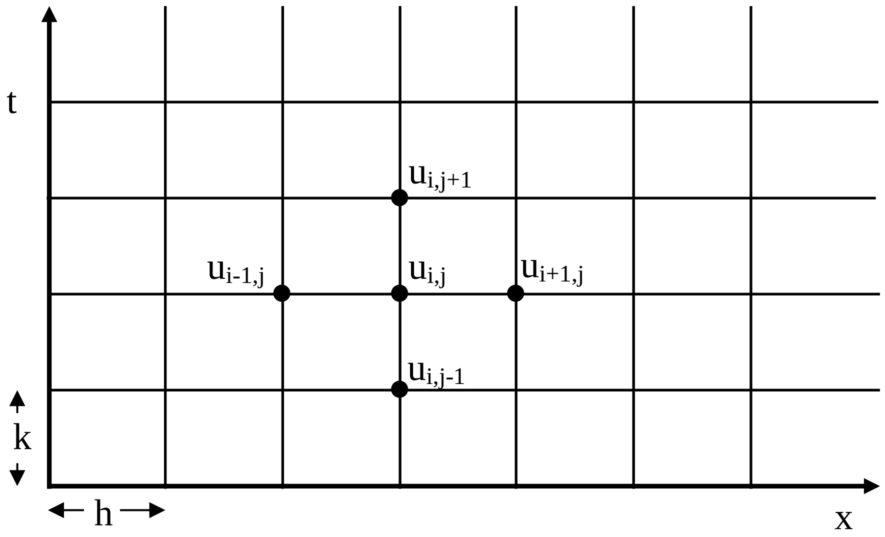


Fig (4.1) The time-space grid

Exercises

Q1. From the general equation (4.1) derive each of advection, wave, and diffusion equations by making some suitable assumptions. Then classify these equations according to the type.

Q2. Why the determination of the type of PDE is important?

Q3. Speaking about Taylor expansion, what do we mean by truncation error of the order $O(h^2)$?

Q4. Derive the formula of second order derivative.

MATLAB Work

Try writing a matlab script to solve equation 4.14. (Use internet)

Homework

1. Rewrite the equations 4.9-4.14 with respect to the time (t) coordinate.
2. Classify the equations 4.6-4.8 to their types.