

Further,  $r' = 19 = 16 + 2 + 1$  and

$$\begin{aligned} 26 &\equiv -3 \pmod{29}, \\ \Rightarrow 26^2 &\equiv 9 \pmod{29}, \\ \Rightarrow 26^4 &\equiv 81 \pmod{29} \equiv -6 \pmod{29}, \\ \Rightarrow 26^8 &\equiv 36 \pmod{29} \equiv 7 \pmod{29}, \\ \Rightarrow 26^{16} &\equiv 49 \pmod{29} \equiv -9 \pmod{29}. \end{aligned}$$

Hence for  $y = 26$

$$\begin{aligned} y^{r'} &= 26^{19} = 26^{16} \cdot 26^2 \cdot 26 \equiv -9 \cdot 9 \cdot -3 \pmod{29} \equiv 27 \cdot 9 \pmod{29} \equiv -2 \cdot 9 \pmod{29} \\ &\equiv -18 \pmod{29} \equiv 11 \pmod{29}, \end{aligned}$$

and so  $x = 11$ .

- (2) Consider  $p = 2633$  and  $r = 29$ . Then  $(r, p - 1) = (29, 2632) = 1$ ,  $m = 2$  and  $r' = 2269$ . Hence we have that

$$\begin{array}{l} \text{T H I S E X A M P L E} \\ \rightarrow \text{T H I S E X A M P L E X} \\ \rightarrow 1907 \mid 0818 \mid 0423 \mid 0012 \mid 1511 \mid 0423 \\ \rightarrow 2199 \mid 1745 \mid 2437 \mid 2425 \mid 1729 \mid 2437 \end{array}$$

We have that  $r = 29 = 16 + 8 + 4 + 1$ . For  $x = 12$ ,

$$\begin{aligned} 12^2 &= 144, \\ 12^4 &= 20736 \equiv -328 \pmod{2633}, \\ \Rightarrow 12^8 &\equiv 107584 \pmod{2633} \equiv -369 \pmod{2633}, \\ \Rightarrow 12^{16} &\equiv 136161 \pmod{2633} \equiv -755 \pmod{2633}. \end{aligned}$$

Hence

$$\begin{aligned} x^r &= 12^{29} = 12^{16} \cdot 12^8 \cdot 12^4 \cdot 12 \equiv -755 \cdot -369 \cdot -328 \cdot 12 \pmod{2633} \\ &\equiv -503 \cdot 1330 \pmod{2633} \equiv 2425 \pmod{2633}, \end{aligned}$$