# Chapter 15

# Pythagorean Triples

**Definition 15.1.** A *Pythagorean Triple* is a set of integers $x, y, z$ satisfying $x^2 + y^2 = z^2$. A *Primitive Pythagorean Triple (PPT)* also has $\gcd(x, y, z) = 1$.

**Lemma 15.2.** *In any PPT $x, y, z$ where $x < y < z$, one of $x, y$ is even, the other odd; $z$ is always odd, e.g. $\{3, 4, 5\}$, $\{7, 24, 25\}$.*

**Proof** If $x, y$ are both even, then so is $x^2 + y^2$, and thus so is $z$. Therefore, the set $x, y, z$ has a common factor of 2, and is not a PPT.

If $x, y$ are both odd, then $x^2 = y^2 = 1 (\text{mod } 4)$. Therefore $z^2 = 2 (\text{mod } 4)$. But for any integer $z$, $z^2 = 0$ or $1$ (mod 4). Therefore $x^2 + y^2 \neq z^2$ if $x, y$ are both odd.

This leaves the only possibility that one of $x, y$ is odd, the other even. In this case, $x^2 + y^2 = 0 + 1 = 1(\text{mod} 4)$, and so $z$ must be odd. □

**Theorem 15.3.** *$x, y, z$ is a PPT, with even $x$, iff $x = 2st, y = s^2 - t^2, z = s^2 + t^2$, with $s > t > 0$, $gcd(s, t) = 1$ and $s \neq t(mod\ 2)$.*

**Proof** Clearly, $x, y, z$ as given is a PT, since $(2st)^2 + (s^2 - t^2)^2 = (s^2 + t^2)^2$. To show that it is primitive, suppose $\gcd(x, y, z) = d > 1$, and take $p$ to be a prime divisor of $d$. Then $p \neq 2$, since $z$ is odd, since it is the sum of an odd square and an even square. Since $p \mid y$ and $p \mid z$, then $p \mid (z + y)$ and $p \mid (z - y)$; that is, $p \mid 2s^2$ and $p \mid 2t^2$. But, since $p \neq 2$, p divides $s$ and $t$. Hence $\gcd(s, t) \neq 1$. Thus $d = 1$ and $x, y, z$ are co-prime, forming a PPT.

To show the converse, first notice that since $\gcd(x, y, z) = 1$, then

$$\gcd(x, y) = \gcd(y, z) = \gcd(x, z) = 1,$$

since if any pair are not coprime, then the triple is not coprime either. Then, if $x, y, z$ is a PPT, with even $x$, $y, z$ are both odd, so $z - y, z + y$ are both even, say $z - y = 2u, z + y = 2v$ (so $y = v - u, z = v + u$). Then the equation $x^2 + y^2 = z^2$ becomes

$$x^2 = z^2 - y^2 = (z - y)(z + y) = 4uv.$$