Chapter 13

The Euler Totient Function [Euler 1707-1783]

Definition 13.1. The *Euler Totient Function* is the mapping $\varphi : \mathbb{N} \mapsto \mathbb{N}$ given by

$$\varphi(n) = |\{m \in \mathbb{N} : 1 \le m \le n, (m, n) = 1\}|.$$

Remark 13.2. We have that

n	1	2	3	4	5	6	7	8	$\cdots$
$\varphi(n)$	1	1	2	2	4	2	6	4	$\cdots$

Theorem 13.3. (i) *For a prime $p \in \mathbb{N}$, $\varphi(p) = p - 1$.*

(ii) *For a prime power p^d (with $p, d \in \mathbb{N}$), $\varphi(p^d) = p^d - p^{d-1}$.*

(iii) *If $m, n \in \mathbb{N}$ satisfy $(m, n) = 1$, then*

$$\varphi(mn) = \varphi(m)\varphi(n).$$

(iv) *If $n \in \mathbb{N}$ has canonical decomposition*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r},$$

then

$$\varphi(n) = \prod_{i=1}^{r} \left(p_i^{\alpha_i} - p_i^{\alpha_i - 1}\right).$$

Proof (i) We have that

$$(1, p) = (2, p) = \dots = (p-1, p) = 1, \quad (p, p) = p.$$

Hence $\varphi(p) = p - 1$.