The equivalence classes are

$$\begin{aligned}
&\overline{0} = \{0,\ \pm n,\ \pm 2n,\ \pm 3n,\ \ldots\};\\
&\overline{1} = \{1,\ \pm n+1,\ \pm 2n+1,\ \pm 3n+1,\ \ldots\};\\
&\overline{2} = \{2,\ \pm n+2,\ \pm 2n+2,\ \pm 3n+2,\ \ldots\};\\
&\vdots\\
&\overline{n-1} = \{n-1,\ \pm 2n-1,\ \pm 3n-1,\ \pm 4n-1,\ \ldots\}.
\end{aligned}$$

The set of all equivalence classes is

$$\mathbb{Z}_n = \left\{\overline{0},\ \overline{1},\ \overline{2},\ \ldots,\ \overline{n-1}\right\}.$$

Basic Properties of Congruences

Recall that for $n \in \mathbb{N}$ and $x,\ y \in \mathbb{Z}$, we say that $x \equiv y \pmod{n}$ (i.e. x is congruent to y modulo n) if $n \mid x - y$.

Ithe next result was shown Example 12.4 (3).

Lemma 12.5. *Let $n \in \mathbb{N}$. Then*

(i) *for any $x \in \mathbb{Z}$, $x \equiv x \pmod{n}$;*

(ii) *for any $x,\ y \in \mathbb{Z}$, $x \equiv y \pmod{n} \Rightarrow y \equiv x \pmod{n}$;*

(iii) *for any $x,\ y,\ z \in \mathbb{Z}$, $x \equiv y \pmod{n}$, $y \equiv z \pmod{n} \Rightarrow x \equiv z \pmod{n}$.*

Lemma 12.6. *Suppose that $n \in \mathbb{N}$ and $x_1,\ y_1,\ x_2,\ y_2 \in \mathbb{Z}$ satisfy $x_1 \equiv y_1 \pmod{n}$ and $x_2 \equiv y_2 \pmod{n}$. Then*

(i) *for any $\lambda_1,\ \lambda_2 \in \mathbb{Z}$, $\lambda_1 x_1 + \lambda_2 x_2 \equiv \lambda_1 y_1 + \lambda_2 y_2 \pmod{n}$;*

(ii) $x_1 x_2 \equiv y_1 y_2 \pmod{n}$.

Proof Since $n \mid x_1 - y_1$, $\exists k_1 \in \mathbb{Z}$ such that $x_1 - y_1 = k_1 n$. Hence $x_1 = y_1 + k_1 n$. Similarly, $\exists k_2 \in \mathbb{Z}$ such that $x_2 = y_2 + k_2 n$.

(i) We have that

$$\lambda_1 x_1 + \lambda_2 x_2 = \lambda_1 (y_1 + k_1 n) + \lambda_2 (y_2 + k_2 n) = \lambda_1 y_1 + \lambda_2 y_2 + (\lambda_1 k_1 + \lambda_2 k_2)\, n.$$

Hence $n \mid (\lambda_1 x_1 + \lambda_2 x_2) - (\lambda_1 y_1 + \lambda_2 y_2)$, giving that

$$\lambda_1 x_1 + \lambda_2 x_2 \equiv \lambda_1 y_1 + \lambda_2 y_2 \pmod{n}.$$