(b) Suppose that $x \not\sim y$.

We argue by contradiction. Indeed, assume that $\overline{x} \cap \overline{y} \neq \emptyset$, and take $t \in \overline{x} \cap \overline{y}$. Since $t \in \overline{x}$, $t \sim x$. Hence $x \sim t$.

Since $t \in \overline{y}$, $t \sim y$. Thus $x \sim t$ and $t \sim y$, giving that $x \sim y$. This is a contradiction.

(c) This follows from (a) and (b).

□

Notation 12.3. Let S be a set and $R \subset S \times S$ be an equivalence relation. We denote by $S/\sim$ or S/R the set of all the equivalence classes.

Examples 12.4. (1) Take $S = \{$ students at Sussex $\}$.

We say that $x \sim y$ if x is a student in the same school at y. Then $\sim$ is indeed an equivalence relation. Further, each element of the set S/R is the set of students in a particular school.

(2) Take $S = \{$ people on Earth $\}$.

We say that $x \sim y$ if x lives in the same country as y. Then $\sim$ is indeed an equivalence relation. Further, each element of the set S/R is the set of all people living in a particular country.

(3) Take $S = \mathbb{Z}$, and pick $n \in \mathbb{N}$.

We say that $x \sim y$ if $n \,|\, x - y$.

(i) Pick $x \in \mathbb{Z}$. Since $n \,|\, 0$, $n \,|\, x - x$ and hence $x \sim x$.

(ii) Suppose that $x, y \in \mathbb{Z}$ and that $x \sim y$. Then $n \,|\, x - y$, and hence $\exists k \in \mathbb{Z}$ such that $x - y = nk$. So $y - x = n\,(-k)$, giving that $n \,|\, y - x$ and hence that $y \sim x$.

(iii) Suppose that $x, y, z \in \mathbb{Z}$, and that $x \sim y$ and $y \sim z$. Then $n \,|\, x - y$ and $n \,|\, y - z$. Hence $\exists k, l \in \mathbb{Z}$ such that $x - y = nk$ and $y - z = nl$. Hence

$$x - z = (x - y) + (y - z) = nk + nl = n\,(k + l)\,,$$

giving that $n \,|\, x - z$, and hence that $x \sim z$.

Instead of using the notation $x \sim y$, from now on we will say that $x \equiv y \pmod{n}$ (i.e. x is congruent to y modulo n) if $n \,|\, x - y$.