Chapter 12

Congruences[Gauss 1777-1855]

Definition 12.1. Let S be a set and $R \subset S \times S$. We introduce the relation $\sim$ defined by

$$x \sim y \quad \Leftrightarrow \quad (x, y) \in R.$$

R is an *equivalence relation* if

(1) $x \sim x \, \forall x$;

(2) $x \sim y \Rightarrow y \sim x$;

(3) $x \sim y,\ y \sim z \Rightarrow x \sim z$.

The *equivalence class of* x is defined to be $\overline{x} = \{t \in S : t \sim x\}$.

Lemma 12.2. *Let S be a set and $R \subset S \times S$ be an equivalence relation.*

(a) *If $x \sim y$, then $\overline{x} = \overline{y}$.*

(b) *If $x \not\sim y$, then $\overline{x} \cap \overline{y} = \emptyset$.*

(c) $S = \bigcap_{x \in S} \overline{x} =$ *disjoint union of equivalence classes.*

Proof (a) Suppose that $x \sim y$.

Take $t \in \overline{x}$. Then $t \sim x$. Since $x \sim y$, it follows that $t \sim y$. Hence $t \in \overline{y}$. So $\overline{x} \subset \overline{y}$.

Take $t \in \overline{y}$. Then $t \sim y$. Since $x \sim y$, $y \sim x$. It follows that $t \sim x$. Hence $t \in \overline{x}$. So $\overline{y} \subset \overline{x}$.

Hence $\overline{x} \subset \overline{y}$ and $\overline{y} \subset \overline{x}$. So $\overline{x} = \overline{y}$, as required.