Theorem 11.19. *If $\zeta(s) = 0$, then $s = 1/2 + iy$.*

Remark 11.20. This is the famous Riemann Hypothesis.

Proof This is not known. There is a \$ 1,000,000 prize for a successful proof (this is one of the seven Clay Millennium Problems). □

We can also use the ζ function to find the probability that two integers, chosen at random, are coprime: the probability that they are both divisible by 2 is $1/2^2$, that they are both divisible by 3 is $1/3^2$ and so on. Therefore the probability that they are not both divisible by $2, 3, 5, \ldots$ is

$$(1 - 1/2^2)(1 - 1/3^2)(1 - 1/5^2)\cdots = 1/\zeta(2) = 6/\pi^2 = 0.6079\ldots.$$

Remark 11.21. Fermat's Factorization Method

To see if a given number is actually prime or not, an efficient method of factorisation was found by Fermat. In order to factorise an odd integer n, suppose $n = ab$. We can write $a = x - y, b = x + y$ with x, y of mixed parity, since a, b are both odd. Then $n = x^2 - y^2$, or $x^2 - n = y^2$. Choose the smallest p such that $p^2 > n$ and consider $p^2 - n, (p + 1)^2 - n, \ldots.$ until a perfect square, q^2 is obtained, with $m^2 - n = q^2$. Then $n = (m - q)(m + q)$.

For example, to factorise $429 : 21^2 - 429 = 12, 22^2 - 429 = 55, 23^2 - 429 = 100 = 10^2$. Thus, $429 = (23 - 10)(23 + 10) = 13 \times 33$.