The Zeta Function

Definition 11.18. Euler defined the zeta function $\zeta(s) = 1 + 1/2^s + 1/3^s + ... = \sum_1^\infty 1/n^s$.

Clearly $\zeta(1) = 1 + 1/2 + 1/3 + ..$ diverges (the harmonic series). Also $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$.

We can rewrite $\zeta(s)$ as follows: since

$$1/2^s\zeta(s) = 1/2^s + 1/4^s + 1/6^s + \cdots,$$

then

$$(1 - 1/2^s)\zeta(s) = 1 + 1/3^s + 1/5^s + \cdots.$$

Similarly

$$(1 - 1/3^s)(1 - 1/2^s)\zeta(s) = 1 + 1/5^s + 1/7^s + \cdots.$$

Carrying on, in exactly the same manner as Eratosthenes's sieve,we find

$$(1 - 1/2^s)(1 - 1/3^s)(1 - 1/5^s)\cdots\zeta(s) = 1;$$

that is,

$$\zeta(s) = \prod_p \frac{1}{1 - 1/p^s} = \prod_p \frac{p^s}{p^s - 1}.$$

An alternative derivation of the product formula for $\zeta(s)$ is to write

$$\begin{aligned}\prod_{All p} \frac{p^s}{p^s - 1} &= \prod_{All p} \frac{1}{1 - 1/p^s} \\ &= \prod_{All p} (1 - 1/p^s)^{-1} \\ &= \prod_{All p} (1 + 1/p^s + (1/p^s)^2 + \cdots);\end{aligned}$$

that is,

$$\begin{aligned}(1 + 1/2^s + (1/2^s)^2 + \cdots)(1 + 1/3^s + (1/3^s)^2 + \cdots)(1 + 1/5^s + (1/5^s)^2 + \cdots)\cdots \\ = 1 + 1/2^s + 1/3^s + 1/4^s + \cdots \\ = \zeta(s),\end{aligned}$$

using the fundamental theorem of arithmetic.

This incidentally gives another proof of the infinitude of primes: since we know $\zeta(1)$ diverges, then $\prod_{All p} \frac{p^1}{p^1-1}$ diverges; that is, $\frac{2}{2-1}\frac{3}{3-1}......\frac{p}{p-1}....$ diverges. If there are only a finite number of primes, then this expression would not diverge. Therefore, by contradiction, there is no highest prime.