(iii) Suppose first of all that $(a, b) = 1$.

We need to show that $[a, b] = ab$. Since $a \mid [a, b]$, there exists $k \in \mathbb{N}$ such that $[a, b] = ka$. Furthermore, since $b \mid [a, b]$, $b \mid ka$.

Since $(a, b) = 1$, Corollary 10.12 gives that $b \mid k$. Hence there exists $k' \in \mathbb{N}$ such that $k = k'b$. Hence $[a, b] = ka = k'ab$. It follows that $ab \mid [a, b]$.

Since $a \mid ab$ and $b \mid ab$, $[a, b] \mid ab$.

Hence $ab \mid [a, b]$ and $[a, b] \mid ab$, which gives that $[a, b] = ab$.

Now suppose that $d \in \mathbb{N}$ satisfies $(a, b) = d$. By Corollary 10.10, it follows that $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$. Hence

$$\left[\frac{a}{d}, \frac{b}{d}\right] = \frac{a}{d} \cdot \frac{b}{d} = \frac{ab}{d^2}.$$

Furthermore, by (ii),

$$[a, b] = d\left[\frac{a}{d}, \frac{b}{d}\right].$$

Hence

$$[a, b] = d \cdot \frac{ab}{d^2} = \frac{ab}{d} = \frac{ab}{(a, b)}.$$

It follows that

$$(a, b)\,[a, b] = ab.$$

□

Example 10.17. Since $(13, 14) = 1$, we have that

$$[13, 14] = 13 \cdot 14 = 182,$$

and hence

$$[65, 70] = [5 \cdot 13, 5 \cdot 14] = 5\,[13, 14] = 5 \cdot 182 = 910,$$

and

$$[130, 140] = [2 \cdot 65, 2 \cdot 70] = 2\,[65, 70] = 2 \cdot 910 = 1820.$$

Alternatively, from Example 10.11, $(65, 70) = 5$ and

$$(130, 140) = (2 \cdot 65, 2 \cdot 70) = 2\,(65, 70) = 2 \cdot 5 = 10.$$

It follows that

$$[65, 70] = \frac{65 \cdot 70}{(65, 70)} = \frac{4550}{5} = 910,$$

and

$$[130, 140] = \frac{130 \cdot 140}{(130, 140)} = \frac{18200}{10} = 1820.$$