Hence $a > b > r_1 > r_2 > r_3 > \ldots > r_{i+1} > r_{i+2} > \ldots \geq 0$.
It follows that $\exists n \in \mathbb{N}$ such that

$$\begin{aligned} r_{n-1} &= q_{n+1} r_n + r_{n+1}, \quad 0 \leq r_{n+1} < r_n, \\ r_n &= q_{n+2} r_{n+1}, \end{aligned}$$

with $r_{n+1} \neq 0$. Take $d = r_{n+1}$. Clearly, $d \in \mathbb{N}$.
Also, $r_n = q_{n+2} d$ and hence $d \,|\, r_n$.
Furthermore

$$r_{n-1} = q_{n+1} r_n + r_{n+1} = q_{n+1} q_{n+2} d + d = (q_{n+1} q_{n+2} + 1)\, d,$$

giving that $d \,|\, r_{n-1}$.
Similarly, $d \,|\, r_{n-2}$, $d \,|\, r_{n-3}$, $\ldots$, $d \,|\, r_1$, $d \,|\, b$, $d \,|\, a$.
So d satisfies (D1) in Definition 10.4.
Suppose now that $e \,|\, a$ and $e \,|\, b$. Then, by Lemma 10.3,

$$\begin{aligned} r_1 &= a - q_1 b & &\Rightarrow e \,|\, r_1; \\ r_2 &= b - q_2 r_1 & &\Rightarrow e \,|\, r_2; \\ r_3 &= r_1 - q_3 r_2 & &\Rightarrow e \,|\, r_3; \\ &\vdots & &\vdots \\ r_n &= r_{n-2} - q_n r_{n-1} & &\Rightarrow e \,|\, r_n; \\ r_{n+1} &= r_{n-1} - q_{n+1} r_n & &\Rightarrow e \,|\, r_{n+1}. \end{aligned}$$

But $r_{n+1} = d$, giving that $e \,|\, d$. So d satisfies (D2) in Definition 10.4.

(ii) Suppose that d, $d' \in \mathbb{N}$ satisfy (D1) and (D2) in Definition 10.4.
Then $d \,|\, d'$ and $d' \,|\, d$. It follows from Remark 10.2 that $d \leq d'$ and $d' \leq d$. Hence $d = d'$.

(iii) From (i),

$$\begin{aligned} r_1 &= a - b q_1, \\ r_2 &= b - q_2 r_1 = b - q_2 (a - b q_1) = -a q_2 + b (1 + q_1 q_2). \end{aligned}$$

We now argue by induction. We prove that for any $i \geq 2$, if there exist integers x_{i-1}, y_{i-1}, x_i, $y_i \in \mathbb{Z}$ such that

$$\begin{aligned} r_{i-1} &= a x_{i-1} + b y_{i-1}, \\ r_i &= a x_i + b y_i; \end{aligned}$$