Chapter 9

Background

As usual, we denote the natural numbers and the integers by $\mathbb{N}$ and $\mathbb{Z}$ respectively:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\},$$
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

We also define

$$\overline{\mathbb{N}} := \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, 4, \ldots\}.$$

Clearly, we have that

$$\mathbb{N} \subset \overline{\mathbb{N}} \subset \mathbb{Z}.$$

We note that addition, multiplication, subtraction and order are defined in $\mathbb{Z}$ and hence can be defined appropriately in $\mathbb{N}$.

A1: Well-Ordering Principle Every non-empty subset S of $\mathbb{N}$ contains a least element, i.e. $\exists a \in S$ such that

$$a \leq x \quad \forall x \in S.$$

A2: Archimedean Property For $a, b \in \mathbb{N}$, $\exists n \in \mathbb{N}$ such that $na \geq b$.

A3: Principle of (Finite) Induction If S is a subset of $\mathbb{N}$ such that

$$\text{(i) } 1 \in S, \quad \text{(ii) } k \in S \Rightarrow k+1 \in S,$$

then $S = \mathbb{N}$.

Remark 9.1. We have that A1 $\Rightarrow$ A2, and that A1 $\Rightarrow$ A3.