Remark 8.16. Above, we factorized a homomorphism through an inclusion homomorphism into a surjective homomorphism followed by an injective homomorphism.

Here we use a quotient homomorphism to again factorize a homomorphism into a surjective homomorphism followed by an injective homomorphism.

Theorem 8.17 (First Isomorphism Theorem)**.** *Let G and H be groups, and $\varphi : G \mapsto H$ be a homomorphism. Then there exists a canonical isomorphism*

$$G/Ker\ \varphi \tilde{\to} Im\varphi.$$

Proof Consider the injective homomorphism

$$\begin{aligned} \psi : G/\text{Ker } \varphi &\mapsto H \\ (\text{Ker } \varphi)\, g &\mapsto \varphi(g) \end{aligned}$$

introduced in Proposition 8.15.

The image of this homomorphism in the group H is clearly $\text{Im}\varphi$.

Hence, by Proposition 7.18, $\psi : G/\text{Ker } \varphi \mapsto H$ can be factorized through the inclusion of the subgroup $\text{Im}\varphi$ in the group H by a homomorphism

$$\Psi : G/\text{Ker } \varphi \mapsto \text{Im}\varphi$$

which is surjective.

Since $\psi : G/\text{Ker } \varphi \mapsto H$ is injective, so too is $\Psi : G/\text{Ker } \varphi \mapsto \text{Im}\varphi$.

Hence $\Psi : G/\text{Ker } \varphi \mapsto \text{Im}\varphi$ is a bijection.

It follows by Proposition 7.9 that it is an isomorphism. □

Corollary 8.18. *Let G and H be finite groups, and $\varphi : G \mapsto H$ be a homomorphism. Then we have that*

$$o(G) = o(Ker\ \varphi)\, o(Im\varphi).$$

Proof By the First Isomorphism Theorem,

$$o(G/\text{Ker } \varphi) = o(\text{Im}\varphi).$$

Furthermore, from Remark 8.10,

$$o(G/\text{Ker } \varphi) = \frac{o(G)}{o(\text{Ker } \varphi)}.$$

Hence

$$\frac{o(G)}{o(\text{Ker } \varphi)} = o(\text{Im}\varphi) \quad \Rightarrow \quad o(G) = o(\text{Ker } \varphi)\, o(\text{Im}\varphi).$$

□