then

$$\varphi(g) = \varphi(g').$$

Suppose that $g,\ g' \in G$ satisfy $(\text{Ker } \varphi)\, g = (\text{Ker } \varphi)\, g'$.

We have that

$$\begin{aligned}
(\text{Ker } \varphi)\, g = (\text{Ker } \varphi)\, g' &\Leftrightarrow g\,(g')^{-1} \in \text{Ker } \varphi && \text{from Remark 8.9} \\
&\Leftrightarrow \varphi\left(g\,(g')^{-1}\right) = 1_H && \text{by definition of Ker } \varphi \\
&\Leftrightarrow \varphi(g)\,\varphi\left((g')^{-1}\right) = 1_H && \text{since } \varphi : G \mapsto H \text{ is a homomorphism} \\
&\Leftrightarrow \varphi(g)\,[\varphi(g')]^{-1} = 1_H && \text{since } \varphi : G \mapsto H \text{ is a homom orphism} \\
&\Leftrightarrow \varphi(g) = 1_H \varphi(g') && \text{multiplying on right by } \varphi(g') \\
&\Leftrightarrow \varphi(g) = \varphi(g').
\end{aligned}$$

Next, we show that the mapping $\psi : G/\text{Ker } \varphi \mapsto H$ is a homomorphism. Pick $g,\ g' \in G$. We have that

$$\begin{aligned}
\psi\left([(\text{Ker } \varphi)\, g]\,[(\text{Ker } \varphi)\, g']\right) &= \psi\left((\text{Ker } \varphi)\, gg'\right) \\
&= \varphi(gg') \\
&= \varphi(g)\,\varphi(g') \\
&\qquad \text{since } \varphi : G \mapsto H \text{ is a homomorphism} \\
&= \psi\left((\text{Ker } \varphi)\, g\right)\psi\left((\text{Ker } \varphi)\, g'\right).
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
\psi\left([(\text{Ker } \varphi)\, g]^{-1}\right) &= \psi\left((\text{Ker } \varphi)\, g^{-1}\right) \\
&= \varphi\left(g^{-1}\right) \\
&= [\varphi(g)]^{-1} \\
&\qquad \text{since } \varphi : G \mapsto H \text{ is a homomorphism} \\
&= [\psi\left((\text{Ker } \varphi)\, g\right)]^{-1},
\end{aligned}$$

and

$$\begin{aligned}
\psi\left(1_{G/\text{Ker } \varphi}\right) &= \psi\left((\text{Ker } \varphi)\, 1_G\right) \\
&= \varphi(1_G) \\
&= 1_H \qquad \text{since } \varphi : G \mapsto H \text{ is a homomorphism.}
\end{aligned}$$

It remains to show that the mapping $\psi : G/\text{Ker } \varphi \mapsto H$ is injective.

From above,

$$\varphi(g) = \varphi(g') \Rightarrow (\text{Ker } \varphi)\, g = (\text{Ker } \varphi)\, g'$$

and hence $\psi : G/\text{Ker } \varphi \mapsto H$ is injective. □