From Remark 8.9, we have that for $g \in G$,

$$Ng = N1_G \quad \Leftrightarrow \quad g1_G^{-1} \in N \quad \Leftrightarrow \quad g \in N.$$

Hence Ker $\chi = N$. □

Remark 8.13. The quotient homomorphism

$$\chi : G \mapsto G/N$$

is surjective.

Example 8.14 (Example 3.14 revisited)**.** Consider the subgroup $4\mathbb{Z} = \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$ of the group $\mathbb{Z}$ of integers under addition. This subgroup is normal, since $\mathbb{Z}$ is an Abelian group under addition. The elements of $\mathbb{Z}/4\mathbb{Z}$ are the right cosets of $4\mathbb{Z}$ in $\mathbb{Z}$:

$$\begin{aligned} 4\mathbb{Z} + 0 &= \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}, \\ 4\mathbb{Z} + 1 &= \{\ldots, -11, -7, -3, 1, 5, 9, 13, \ldots\}, \\ 4\mathbb{Z} + 2 &= \{\ldots, -10, -6, -2, 2, 6, 10, 14, \ldots\}, \\ 4\mathbb{Z} + 3 &= \{\ldots, -9, -5, -1, 3, 7, 11, 15, \ldots\}. \end{aligned}$$

These cosets partition $\mathbb{Z}$.

The group operations of addition, negation and zero on $\mathbb{Z}/4\mathbb{Z}$ are given by, respectively,

$$\begin{aligned} &(k + 4\mathbb{Z}) + (k' + 4\mathbb{Z}) = (k + k') + 4\mathbb{Z}, \\ &-(k + 4\mathbb{Z}) = (-k) + 4\mathbb{Z}, \\ &0 + 4\mathbb{Z} = 1_{\mathbb{Z}/4\mathbb{Z}}. \end{aligned}$$

Proposition 8.15. *Let G and H be groups, and $\varphi : G \mapsto H$ be a homomorphism. Then φ may be factorized uniquely through the quotient homomorphism from G to the quotient group $G/Ker\ \varphi$ of G by the normal subgroup $Ker\ \varphi$.*

Furthermore, the mapping

$$\begin{aligned} \psi : G/Ker\ \varphi &\mapsto H \\ (Ker\ \varphi)\, g &\mapsto \varphi(g). \end{aligned}$$

is an injective homomorphism.

Proof First of all, we need to show that the mapping $\psi : G/\text{Ker } \varphi \mapsto H$ is well-defined, i.e. that if $g, g' \in G$ satisfy

$$(\text{Ker } \varphi)\, g = (\text{Ker } \varphi)\, g'$$