Finally, we verify the identity axiom. Pick $g \in G$. We have that

$$(Ng)\, 1_{G/N} = (Ng)\,(N1_G) = Ng1_G = Ng,$$
$$1_{G/N}\,(Ng) = (N1_G)\,(Ng) = N1_Gg = Ng.$$

□

Definition 8.8. Let G be a group and N be a normal subgroup of G. The group G/N of the right cosets of N in G is called the *quotient group*.

Remark 8.9. In G/N, we have that

$$Ng = Nh \quad \Leftrightarrow \quad gh^{-1} \in N.$$

Remark 8.10. The order of the group G/N is the number of right cosets of N in G. Hence, from Corollary 3.17,

$$o\,(G/N) = \frac{o\,(G)}{o\,(N)}.$$

Definition 8.11. Let G be a group and N be a normal subgroup of G. The *quotient homomorphism*

$$\chi\,:\, G \mapsto G/N$$

from G to the quotient group G/N is the map which assigns to each $g \in G$ its right coset $Ng \in G/N$:

$$\chi\,(g) = Ng \in G/N \quad \forall g \in G.$$

Proposition 8.12. *Let G be a group and N be a normal subgroup of G. The quotient homomorphism $\chi\,:\, G \mapsto G/N$ is indeed a homomorphism with kernel N.*

Proof That $\chi\,:\, G \mapsto G/N$ is a homomorphism is a consequence of the way we define the operations on G/N.

Indeed, pick $g,\, g' \in G$. Then we have that

$$\chi\,(gg') = Ngg' = (Ng)\,(Ng') = \chi\,(g)\,\chi\,(g')\,.$$

Further, we have that

$$\chi\left(g^{-1}\right) = Ng^{-1} = (Ng)^{-1} = [\chi\,(g)]^{-1}\,,$$

and

$$\chi\,(1_G) = N1_G = 1_{G/N}.$$

Finally, we wish to show that the kernel of χ is N.

Indeed, by definition

$$\text{Ker}\,\chi = \left\{g \in G\,:\,\chi\,(g) = 1_{G/N}\right\} = \{g \in G\,:\, Ng = N1_G\}\,.$$