Indeed, we have that

$$g = 1_G g \in Ng = Nh = \{xh \in G \, : \, x \in N\}.$$

It follows that there exists $x \in N$ such that $g = xh$.

Hence $gh^{-1} = x \in N$.

Similarly, $g' (h')^{-1} = x' \in N$.

Since N is normal,

$$gh^{-1} \in N \quad \Rightarrow \quad h^{-1}g = h^{-1}\left(gh^{-1}\right)h \in N.$$

Since N is closed under the operation of product of the group G,

$$g' (h')^{-1}, \, h^{-1}g \in N \quad \Rightarrow \quad h^{-1}gg' (h')^{-1} \in N.$$

The normality of N gives that

$$N \ni \left(h^{-1}\right)^{-1} h^{-1}gg' (h')^{-1} h^{-1} = hh^{-1}gg' (h')^{-1} h^{-1} = gg' (h')^{-1} h^{-1} = (gg') (hh')^{-1}$$

It follows that $gg' \in Nhh'$, and hence that

$$Ngg' = Nhh'.$$

The proofs that the operations of inverse and identity on G/N are well-defined are similar.

Proposition 8.7. *Let G be a group and N be a normal subgroup of G. The set G/N of the right cosets of N in G is a group under the operations of product, inverse and identity defined above.*

Proof We need to check that the closure, associativity, inverse and identity axioms are satisfied.

The closure axiom is satisfied since for $g, g' \in G$, $(Ng)(Ng') = Ngg' \in G/N$.

We verify the associativity axiom next. Pick $g, g', g'' \in G$. We have that

$$\begin{aligned}
((Ng)(Ng'))(Ng'') &= (Ngg')(Ng'') \\
&= N(gg')g'' \\
&= Ng(g'g'') && \text{by the associativity axiom of } G \\
&= (Ng)(Ng'g'') \\
&= (Ng)((Ng')(Ng'')).
\end{aligned}$$

We verify next the inverse axiom. Pick $g \in G$. We have that

$$\begin{aligned}
(Ng)\left(Ng^{-1}\right) &= Ngg^{-1} = N1_G = 1_{G/N}, \\
\left(Ng^{-1}\right)(Ng) &= Ng^{-1}g = N1_G = 1_{G/N}.
\end{aligned}$$