(3) Consider the subgroup A_n of the group S_n of permutations of degree n which consists of the even such permutations.

Pick $\sigma \in A_n$ and $\tau \in S_n$. We have that $(-1)^\sigma = +1$. Furthermore, $(-1)^{\tau^{-1}} = (-1)^\tau$ and hence

$$(-1)^{\tau^{-1}\sigma\tau} = (-1)^{\tau^{-1}}(-1)^\sigma(-1)^\tau = [(-1)^\tau]^2 = 1 \quad \Rightarrow \quad \tau^{-1}\sigma\tau \in A_n.$$

Hence A_n *is* a normal subgroup of S_n.

(4) Consider the subgroup $H = \{i, (1\,2)\}$ of the group S_3 of permutations of degree 3. We have that

$$(1\,3)^{-1}(1\,2)(1\,3) = (3\,1)(1\,2)(1\,3) = (1\,3)(1\,2)(1\,3) = (2\,3) \notin H.$$

Hence H is *not* a normal subgroup of S_3.

Proposition 8.4. *Let G and H be groups, and $\varphi : G \mapsto H$ be a homomorphism. The kernel of φ, Ker φ, is a normal subgroup of G.*

Proof Pick $x \in \text{Ker}\ \varphi$ and $g \in G$. Since $\varphi : G \mapsto H$ is a homomorphism,

$$\varphi\left(g^{-1}xg\right) = \varphi\left(g^{-1}\right)\varphi(x)\varphi(g) = [\varphi(g)]^{-1}1_H\varphi(g) = [\varphi(g)]^{-1}\varphi(g) = 1_H.$$

□

Construction 8.5. Given a normal subgroup N of a group G, consider the set G/N consisting of all right cosets

$$Ng = \{xg \in G : x \in N\}$$

of the subgroup N in the group G. We define operations of product, inverse and identity on G/N by

$$(Ng)(Ng') = Ngg',$$
$$(Ng)^{-1} = Ng^{-1},$$
$$1_{G/N} = N1_G \qquad \text{(the set of elements in } N\text{)}.$$

Remark 8.6. We need to check that the above operations are well-defined.

To show that the operation of product is well-defined, suppose that $g,\ g',\ h,\ h' \in G$ satisfy $Ng = Nh$ and $Ng' = Nh'$. We wish to show that

$$(Ng)(Ng') = (Nh)(Nh'),$$

i.e. that $Ngg' = Nhh'$.