Chapter 8

Normal Subgroups and Quotient Groups

Definition 8.1. A subgroup N of a group G is said to be a *normal subgroup* of G if

$$h \in N \quad \Rightarrow \quad g^{-1}hg \in N \, \forall g \in G.$$

Lemma 8.2. *Let G be an Abelian group, and N be a subgroup of G. Then N is normal.*

Proof Pick $h \in N$. Since G is Abelian,

$$g^{-1}hg = g^{-1}gh = 1_G h = h \in N.$$

□

Examples 8.3. (1) Consider the group $\mathrm{GL}(2, \mathbb{R})$ of invertible 2×2 matrices over the real numbers. Consider the subgroup N of $\mathrm{GL}(2, \mathbb{R})$ which consists of those 2×2 real matrices whose determinants are 1. Pick $A \in N$ and $B \in \mathrm{GL}(2, \mathbb{R})$. Using the properties of determinants gives that

$$\begin{aligned} \det\left(B^{-1}AB\right) &= \det\left(B^{-1}\right)\det(A)\det(B) \\ &= \frac{1}{\det(B)}\det(A)\det(B) \\ &= \det(A) \\ &= 1. \end{aligned}$$

Hence $B^{-1}AB \in N$. It follows that N *is* a normal subgroup of $\mathrm{GL}(2, \mathbb{R})$.

(2) For any $n \in \mathbb{N}$, the subgroup

$$n\mathbb{Z} = \{nm \in \mathbb{Z} : m \in \mathbb{Z}\} = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\}$$

of the group $\mathbb{Z}$ of integers under addition *is* normal, since $\mathbb{Z}$ is an Abelian group under addition.