Construction 7.21. Given any finite group G of order n, consider the mapping from G to the symmetric group S_n obtained by labelling the elements of G as $g_1, g_2, \ldots, g_n$ and defining

$$\begin{aligned} \varphi : G &\mapsto S_n \\ g &\mapsto \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1 g & g_2 g & \cdots & g_n g \end{pmatrix} \end{aligned}$$

Example 7.22. Consider the group $C_3 = \{1, \omega, \omega^2\}$ of cube roots of unity, and take $g_1 = 1$, $g_2 = \omega$, $g_3 = \omega^2$. Then the mapping $\varphi : C_3 \mapsto S_3$ satisfies

$$\begin{aligned} C_3 \ni 1 &\mapsto \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \end{pmatrix}, \text{ i.e.} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}; \\ C_3 \ni \omega &\mapsto \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{pmatrix}, \text{ i.e.} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}; \\ C_3 \ni \omega^2 &\mapsto \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix}, \text{ i.e.} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}. \end{aligned}$$

Corollary 7.23 (Cayley's Theorem). *Any finite group G is isomorphic to a subgroup of the symmetric group S_n, where n is the order of G.*

Proof Consider applying Construction 7.21 to G. Then after labelling the elements of G, we are able to define the mapping $\varphi : G \mapsto S_n$.

It is straightforward to show that $\mathrm{Im}\varphi$ is a subgroup of S_n and that $\varphi : G \mapsto S_n$ is a homomorphism of groups.

To show further that $\varphi : G \mapsto S_n$ is injective, suppose that $g, \tilde{g} \in G$ satisfy

$$\varphi(g) = \varphi(\tilde{g}).$$

Then

$$\begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1 g & g_2 g & \cdots & g_n g \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1 \tilde{g} & g_2 \tilde{g} & \cdots & g_n \tilde{g} \end{pmatrix},$$

i.e.

$$g_i g = g_i \tilde{g} \quad \forall i \in \{1, 2, \ldots, n\}.$$

In particular, we have that

$$1g = 1\tilde{g} \quad \Rightarrow \quad g = \tilde{g}.$$

Hence $\varphi : G \mapsto S_n$ is injective.

It follows from Corollary 7.20 that the canonical homomorphism $\psi : G \mapsto \mathrm{Im}\varphi$ is an isomorphism.

Hence G is isomorphic to $\mathrm{Im}\varphi$, a subgroup of S_n. □