Definition 7.16. For any subgroup H of a group G, by the *inclusion homomorphism* is meant the mapping

$$i : H \mapsto G$$

which assigns ro each $h \in H$ the element $h \in G$.

Remark 7.17. The inclusion homomorphism of a subgroup H of a group G is indeed a homomorphism. Moreover, it is an injective homomorphism.

Proposition 7.18. *Let G and H be groups, and $\varphi : G \mapsto H$ be a homomorphism. Then φ may be factorized through the inclusion of the subgroup* $\mathrm{Im}\varphi$ *in the group H by a homomorphism*

$$\psi : G \mapsto \mathrm{Im}\varphi$$

which is surjective.

Proof Define $\psi : G \mapsto \mathrm{Im}\varphi$ by

$$\psi(g) = \varphi(g) \quad \forall g \in G.$$

Then, since φ is a homomorphism, ψ is a homomorphism. Moreover, ψ is clearly surjective.

Clearly

$$\varphi = i \circ \psi,$$

where $i : \mathrm{Im}\varphi \mapsto$ H is the inclusion homomorphism defined above. □

Remark 7.19. We call $\psi : G \mapsto \mathrm{Im}\varphi$ the *canonical homomorphism* associated with $\varphi : G \mapsto H$. As we have just seen, it is surjective by construction.

Corollary 7.20. *Let G and H be groups, and $\varphi : G \mapsto H$ be a homomorphism. Then the following assertions are equivalent:*

(a) *$\varphi : G \mapsto H$ is injective;*

(b) *the canonical homomorphism $\psi : G \mapsto \mathrm{Im}\varphi$ is an isomorphism.*

Proof (a) $\Rightarrow$ (b) Suppose that (a) holds, i.e. that $\varphi : G \mapsto H$ is injective.

Then the canonical homomorphism $\psi : G \mapsto \mathrm{Im}\varphi$ is also injective. Furthermore, it is surjective by Remark 7.19. Hence it is bijective.

It follows from Proposition 7.9 that $\psi : G \mapsto \mathrm{Im}\varphi$ is an isomorphism.

(b) $\Rightarrow$ (a) Suppose that (b) holds, i.e. that the canonical homomorphism $\psi : G \mapsto \mathrm{Im}\varphi$ is an isomorphism.

It follows from Proposition 7.9 that $\psi : G \mapsto \mathrm{Im}\varphi$ is bijective, and hence it is injective.

Hence $\varphi : G \mapsto H$ is injective. □