Pick $h \in \mathrm{Im}(\varphi)$. To show that $\mathrm{Im}(\varphi)$ satisfies the inverse axiom, we must show that $h^{-1} \in \mathrm{Im}(\varphi)$.

By the definition of $\mathrm{Im}(\varphi)$, there exists $g \in G$ such that

$$h = \varphi(g).$$

Since $\varphi : G \mapsto H$ is a homomorphism, it preserves the group operation of inverse. Hence

$$h^{-1} = [\varphi(g)]^{-1} = \varphi(g^{-1}),$$

giving that $h^{-1} \in \mathrm{Im}(\varphi)$ as required.

Hence $\mathrm{Im}(\varphi)$ is a subgroup of H. □

Definition 7.13. Let G and H be groups and

$$\varphi : G \mapsto H$$

be a homomorphism. By the kernel of φ is meant the subset

$$\mathrm{Ker}\ \varphi = \{g \in G : \varphi(g) = 1_H\}.$$

of the group G.

Proposition 7.14. *Let G and H be groups and $\varphi : G \mapsto H$ be a homomorphism. Then the* kernel *of φ, Ker φ, is a subgroup of G. Moreover, the following assertions are equivalent:*

(a) *$\varphi : G \mapsto H$ is injective;*

(b) *the kernel of φ, Ker φ, is the trivial subgroup $\{1_G\}$ of G.*

Proof By Proposition 3.2, to show that $\mathrm{Ker}(\varphi)$ is a subgroup of G it is sufficient to show that it satisfies the closure and inverse axioms with respect to the product operation of G.

To show that it satisfies the closure axiom, suppose that $g_1, g_2 \in \mathrm{Ker}\ \varphi$:

$$\varphi(g_1) = \varphi(g_2) = 1_H.$$

We need to show that $g_1 g_2 \in \mathrm{Ker}\ \varphi$.

Indeed, since $\varphi : G \mapsto H$ is a homomorphism,

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) = 1_H \cdot 1_H = 1_H.$$

To show that Ker φ satisfies the inverse axiom, suppose that $g \in \mathrm{Ker}\ \varphi$:

$$\varphi(g) = 1_H.$$

We need to show that $g^{-1} \in \mathrm{Ker}\ \varphi$.