Let $h_1, h_2 \in H$. Since $\varphi : G \mapsto H$ is a homomorphism, it preserves the group operation of product and hence

$$\varphi(\psi(h_1)\psi(h_2)) = \varphi(\psi(h_1))\varphi(\psi(h_2)) = h_1 h_2,$$

where the second equality comes from using that $\varphi \circ \psi = i_H$. It follows that

$$\psi(h_1 h_2) = \psi(\varphi(\psi(h_1)\psi(h_2))) = \psi(h_1)\psi(h_2),$$

where the second equality comes from using that $\psi \circ \varphi = i_G$. □

Remark 7.10. It follows that if there exists an isomorphism from a group G to the group H, then G and H have the same order.

However, the converse is *not* true. For example, C_6 is *not* isomorphic to the group S_3 even though both groups are of order 6.

Definition 7.11. Let G and H be groups and $\varphi : G \mapsto H$ be a homomorphism. The *image* of φ, $\mathrm{Im}(\varphi)$, is the subset

$$\mathrm{Im}(\varphi) = \{\mathrm{h} \in \mathrm{H} : \exists\, \mathrm{g} \in \mathrm{G} \text{ with } \mathrm{h} = \varphi(\mathrm{g})\}$$

of the group H.

Proposition 7.12. *Let G and H be groups and $\varphi : G \mapsto H$ be a homomorphism. The* image *of φ, $\mathrm{Im}(\varphi)$, is a subgroup of H. Moreover, the following are equivalent:*

(i) *$\varphi : G \mapsto H$ is surjective;*

(ii) $\mathrm{Im}(\varphi) = \mathrm{H}$.

Proof Note that $\varphi(i_G) \in \mathrm{Im}(\varphi)$, and hence $\mathrm{Im}(\varphi)$ is a non-empty subset of H.

By Proposition 3.2, to show that $\mathrm{Im}(\varphi)$ is a subgroup of H it is sufficient to show that it satisfies the closure and inverse axioms with respect to the product operation of H.

Pick $h, h' \in \mathrm{Im}(\varphi)$. To show that $\mathrm{Im}(\varphi)$ satisfies the closure axiom, we must show that $hh' \in \mathrm{Im}(\varphi)$.

By the definition of $\mathrm{Im}(\varphi)$, there exist $g, g' \in G$ such that

$$h = \varphi(g) \quad \text{and} \quad h' = \varphi(g').$$

Since $\varphi : G \mapsto H$ is a homomorphism, it preserves the group operation of product. Hence

$$\varphi(gg') = \varphi(g)\varphi(g') = hh',$$

giving that $hh' \in \mathrm{Im}(\varphi)$ as required.