Remark 7.7. It is easy to see that ψ is also an isomorphism with inverse homomorphism φ.

Example 7.8. Let n be a positive integer and take

$$\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right),$$

the primitive n^{th} root of unity. Take $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ to be the group of integers mod n under addition mod n, and C_n to be the group of the n^{th} roots of unity under complex multiplication. From Example 7.3 (4), the mapping

$$\begin{aligned} \varphi: \quad \mathbb{Z}_n &\mapsto C_n \\ m &\mapsto \varphi(m) = \omega^m. \end{aligned}$$

and its inverse

$$\begin{aligned} \psi: \quad C_n &\mapsto \mathbb{Z}_n \\ \omega^m &\mapsto \psi(\omega^m) = m \quad \text{for } m \in \mathbb{Z}_n. \end{aligned}$$

are both homomorphisms. Hence φ and ψ are isomorphisms.

Proposition 7.9. *Let G and H be groups and*

$$\varphi : G \mapsto H$$

be a homomorphism. Then the following are equivalent:

(i) $\varphi : G \mapsto H$ *is an isomorphism;*

(ii) $\varphi : G \mapsto H$ *is bijective.*

Proof Let G and H be groups.

To show that (i) $\Rightarrow$ (ii), suppose that $\varphi : G \mapsto H$ is an isomorphism.

Then φ is an invertible mapping from G to H(with inverse given by the inverse homomorphism $\psi : H \mapsto G$ of φ). Hence $\varphi : G \mapsto H$ is bijective.

To show that the (ii) $\Rightarrow$ (i), suppose that $\varphi : G \mapsto H$ is a bijective mapping. Then it is invertible. So there exists a mapping $\psi : H \mapsto G$ such that

$$\psi(\varphi(g)) = g \quad \forall\, g \in G \qquad \text{and} \qquad \varphi(\psi(h)) = h \quad \forall\, h \in H,$$

i.e.

$$\psi \circ \varphi = i_G \quad \text{and} \quad \varphi \circ \psi = i_H,$$

where i_G and i_H are the identity isomorphisms of G and H respectively.

To show that ψ is a homomorphism, by Remarks 7.2 (3) it is sufficient to show that it preserves the group operation of product.