(3) Let n be a positive integer and take

$$\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right),$$

the primitive n^{th} root of unity. Take $\mathbb{Z}$ to be the group of integers under addition, and C_n to be the group of the n^{th} roots of unity under complex multiplication. Consider the mapping

$$\begin{array}{rccl} \varphi: & \mathbb{Z} & \mapsto & C_n \\ & m & \mapsto & \varphi(m) = \omega^m \end{array}$$

To show that φ is a homomorphism, by Remarks 7.2 (3) it is sufficient to show that it preserves the group operation of product.

Pick $m,\ m' \in \mathbb{Z}$. Then

$$\varphi(m + m') = \omega^{m+m'} = \omega^m \omega^{m'} = \varphi(m)\,\varphi(m'),$$

where the second equality follows from the properties of powers.

(4) Let n be a positive integer and take

$$\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right),$$

the primitive n^{th} root of unity. Take $\mathbb{Z}_n = \{0,\ 1,\ 2,\ \ldots,\ n-1\}$ to be the group of integers mod n under addition mod n, and C_n to be the group of the n^{th} roots of unity under complex multiplication. Consider the mapping

$$\begin{array}{rccl} \varphi: & \mathbb{Z}_n & \mapsto & C_n \\ & m & \mapsto & \varphi(m) = \omega^m. \end{array}$$

To show that φ is a homomorphism, by Remarks 7.2 (3) it is sufficient to show that it preserves the group operation of product.

Pick $m,\ m' \in \mathbb{Z}_n$. Then, since $m +_n m' = m + m' - kn$ for some non-negative integer k,

$$\begin{aligned} \varphi(m +_n m') &= \omega^{m +_n m'} = \omega^{m+m'-kn} = \omega^m \omega^{m'} \omega^{-kn} = \omega^m \omega^{m'} (\omega^n)^{-k} = \omega^m \omega^{m'} 1^{-k} \\ &= \omega^m \omega^{m'} = \varphi(m)\,\varphi(m'), \end{aligned}$$

where the third and fourth equalities follow from the properties of powers.

Consider now the inverse mapping ψ of φ:

$$\begin{array}{rccl} \psi: & C_n & \mapsto & \mathbb{Z}_n \\ & \omega^m & \mapsto & \psi(\omega^m) = m \quad \text{for } m \in \mathbb{Z}_n. \end{array}$$