Applying the Cancellation Law 1.9 in H gives that

$$1_H = \varphi\left(1_G\right),$$

proving (iii). It follows from (2) that (ii) is satisfied.

Example 7.3. (1) Take $\mathrm{GL}\left(n, \mathbb{R}\right)$ to be the group of $n \times n$ invertible matrices over $\mathbb{R}$ under matrix multiplication, and $\mathbb{R}^{\star}$ to be the group of non-zero real numbers under multiplication. Consider the mapping

$$\begin{array}{rrcl} \varphi: & \mathrm{GL}\left(n, \mathbb{R}\right) & \mapsto & \mathbb{R}^{\star} \\ & A & \mapsto & \varphi\left(A\right) = \det\left(A\right) \end{array}$$

which maps an invertible $n \times n$ real matrix A to its determinant $\det\left(A\right)$. Since A is invertible, $\det\left(A\right) \neq 0$. Hence we indeed have that

$$A \in \mathrm{GL}\left(n, \mathbb{R}\right) \quad \Rightarrow \quad \varphi\left(A\right) \in \mathbb{R}^{\star}.$$

To show that φ is a homomorphism, by Remarks 7.2 (3) it is sufficient to show that it preserves the group operation of product.

Pick $A,\ B \in \mathrm{GL}\left(n, \mathbb{R}\right)$. Then

$$\varphi\left(AB\right) = \det\left(AB\right) = \det\left(A\right)\det\left(B\right) = \varphi\left(A\right)\varphi\left(B\right),$$

where the second equality follows from the properties of determinants.

(2) Take $C_2 = \{-1,\ 1\}$ to be the group of the second roots of unity under multiplication. Consider the mapping φ from the group S_n of permutations of degree n to C_2 given by

$$\begin{array}{rrcl} \varphi: & S_n & \mapsto & C_2 \\ & \sigma & \mapsto & \varphi\left(\sigma\right) = \left(-1\right)^{\sigma}. \end{array}$$

So for any permutation σ of degree n, $\varphi\left(\sigma\right)$ is its sign.

To show that φ is a homomorphism, by Remarks 7.2 (3) it is sufficient to show that it preserves the group operation of product.

Pick $\sigma,\ \tilde{\sigma} \in S_n$. Then

$$\varphi\left(\sigma\tilde{\sigma}\right) = \left(-1\right)^{\sigma\tilde{\sigma}} = \left(-1\right)^{\sigma}\left(-1\right)^{\tilde{\sigma}} = \varphi\left(\sigma\right)\varphi\left(\tilde{\sigma}\right),$$

where the second equality follows from Proposition 5.25.