# Chapter 7

# Homomorphisms

**Definition 7.1.** A *homomorphism*

$$\varphi : G \mapsto H$$

from a group $G$ to a group $H$ is a mapping from $G$ to $H$ satisfying

$$\begin{aligned} &\text{(i)} \quad \varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) \quad \forall\, g_1,\, g_2 \in G, \\ &\text{(ii)} \quad \varphi(g^{-1}) = [\varphi(g)]^{-1} \qquad \forall\, g \in G, \\ &\text{(iii)} \; \varphi(1_G) = 1_H. \end{aligned}$$

**Remarks 7.2.** (1) A homomorphism of groups is a mapping which preserves the group operations of product, inverse and identity.

(2) For a mapping to be a homomorphism, it is sufficient for it to preserve the product and identity operations. Indeed, suppose that (i) and (iii) are satisfied and pick $g \in G$. Then

$$\varphi(g^{-1})\varphi(g) = \varphi(g^{-1}g) = \varphi(1_G) = 1_H,$$

where the first, second and third equalities follow from (i), the inverse axiom of $G$ and (iii) respectively. Similarly,

$$\varphi(g)\varphi(g^{-1}) = 1_H,$$

proving (ii).

(3) One can go further and note that for a mapping to be a homomorphism, it is sufficient for it to preserve the product operation. Indeed, suppose that (i) is satisfied. Then

$$1_H \cdot \varphi(1_G) = \varphi(1_G) = \varphi(1_G \cdot 1_G) = \varphi(1_G)\varphi(1_G),$$

where the first, second and third equalities follow from the identity axioms of $H$ and $G$ and (i) respectively.