Chapter 6

Symmetries and Actions

Definition 6.1. For any set X containing n elements, we may consider the set $S(X)$ of invertible mappings from X to itself.

Given any $x \in X$ and any invertible mapping $\varphi \in S(X)$, we may denote the action of the mapping on the element by

$$x \mapsto x\varphi.$$

Given any $\varphi, \psi \in S(X)$, we may define

- the invertible mapping $\varphi\psi$ by

$$x \mapsto (x\varphi)\,\psi \quad \forall x \in X;$$

- the invertible mapping $\varphi^{-1} \in S(X)$ to be the inverse mapping of $\varphi \in S(X)$;
- the identity mapping $i \in S(X)$ by

$$x \mapsto x \quad \forall x \in X.$$

The set $S(X)$ of invertible mappings from X to itself will be called the *symmetric group* S_n of degree n, the group of permutations of $\{1, 2, \cdots, n\}$.

Definition 6.2. We call a subgroup G of the symmetric group $S(X)$ of a set X a *group of symmetries* on the set X.

In particular, given any subgroup G of the symmetric group $S(X)$ and any subset Y of the set X, we may consider the subgroup of G consisting of all $\varphi \in G$ for which

$$x \in Y \quad \Rightarrow \quad x\varphi \in Y.$$

This is the subgroup of G consisting of the mappings under which the subset Y of X is *invariant*.