Consider the mapping from A_n to S_n given by multiplying on the right by the transposition $(1\ 2)$.

By definition, if $\sigma \in A_n$ then it can be expressed as a product of an even number of transpositions. Hence $\sigma\,(1\ 2)$ can be expressed as the product of an odd number of transpositions, and thus is an odd permutation of degree n.

Hence this mapping maps A_n into the set of odd permutations of degree n. It remains to show that it is injective and that its image is the whole of the set of odd permutations of degree n.

To show injectivity, suppose that $\sigma, \tilde{\sigma} \in A_n$ satisfy

$$\sigma\,(1\ 2) = \tilde{\sigma}\,(1\ 2).$$

Multiplying on the right by $(2\ 1)$ gives that

$$\sigma = \sigma\,(1\ 2)\,(2\ 1) = \tilde{\sigma}\,(1\ 2)\,(2\ 1) = \tilde{\sigma}.$$

Hence the mapping is injective.

To show surjectivity, consider an odd permutation σ of degree n. The above mapping maps $\sigma\,(1\ 2)$ to $\sigma\,(1\ 2)\,(1\ 2)$, i.e. to σ. Further, since σ is odd it can be expressed as the product of an odd number of transpositions. Hence $\sigma\,(1\ 2)$ can be expressed as an even number of transpositions and thus is even. So $\sigma\,(1\ 2) \in A_n$ is mapped to σ, giving surjectivity.

Since a bijection exists between the sets of odd and even permutations of degree n, these two sets contain the same number of elements. Since S_n is partitioned into these two sets, we must have that there are $\frac{o(S_n)}{2}$ odd and $\frac{o(S_n)}{2}$ even permutations of degree n, i.e.

$$o\,(A_n) = \frac{o\,(S_n)}{2} = \frac{n!}{2}.$$

□

Remark 5.27. Since $(i_1\ i_2\ i_3\ i_4\ \ldots\ i_k) = (i_1\ i_2)\,(i_1\ i_3)\,(i_1\ i_4)\ldots(i_1\ i_k)$, a cycle of odd length k is an even permutation, and a cycle of even length k is an odd permutation.

Example 5.28 ($n = 3$)**.** The subgroup A_3 of even permutations in the group

$$S_3 = \{i,\ (1\ 2)\,,\ (1\ 3)\,,\ (2\ 3)\,,\ (1\ 2\ 3)\,,\ (1\ 3\ 2)\}\,,$$

where i is the identity permutation, of permutations of degree 3 is

$$A_3 = \{i,\ (1\ 2\ 3)\,,\ (1\ 3\ 2)\}\,.$$