and

$$f_\sigma(x_1, x_2, \ldots, x_n) = (-1)^{k'} f(x_1, x_2, \ldots, x_n) = -f(x_1, x_2, \ldots, x_n).$$

It follows that

$$\begin{aligned} & f(x_1, x_2, \ldots, x_n) = f_\sigma(x_1, x_2, \ldots, x_n) = -f(x_1, x_2, \ldots, x_n) \\ \Rightarrow\ & f(x_1, x_2, \ldots, x_n) \equiv 0, \end{aligned}$$

a contradiction. Hence

$$f_\sigma(x_1, x_2, \ldots, x_n) = (-1)^k f(x_1, x_2, \ldots, x_n) = (-1)^\sigma f(x_1, x_2, \ldots, x_n),$$

giving that

$$f_{\sigma\tilde{\sigma}}(x_1, x_2, \ldots, x_n) = (-1)^{\sigma\tilde{\sigma}} f(x_1, x_2, \ldots, x_n).$$

Furthermore,

$$\begin{aligned} f_{\sigma\tilde{\sigma}}(x_1, x_2, \ldots, x_n) &= (f_\sigma)_{\tilde{\sigma}}(x_1, x_2, \ldots, x_n) \\ &= (-1)^\sigma f_{\tilde{\sigma}}(x_1, x_2, \ldots, x_n) \\ &= (-1)^\sigma (-1)^{\tilde{\sigma}} f(x_1, x_2, \ldots, x_n). \end{aligned}$$

Hence

$$\begin{aligned} & (-1)^{\sigma\tilde{\sigma}} f(x_1, x_2, \ldots, x_n) = f_{\sigma\tilde{\sigma}}(x_1, x_2, \ldots, x_n) = (-1)^\sigma (-1)^{\tilde{\sigma}} f(x_1, x_2, \ldots, x_n) \\ \Rightarrow\ & (-1)^{\sigma\tilde{\sigma}} = (-1)^\sigma (-1)^{\tilde{\sigma}}. \end{aligned}$$

□

Corollary 5.26. *For any positive integer $n \geq 2$, the subset A_n of the group S_n of permutations of degree n which consists of the even such permutations is a subgroup of S_n of order*

$$o(A_n) = \frac{o(S_n)}{2} = \frac{n!}{2}.$$

Proof Pick a positive integer $n \geq 2$. The identity permutation of order n can be expressed as a product of zero transpositions. Hence it is even and thus belongs to A_n.

By Proposition 3.4, we now only need to show that A_n satisfies the closure axiom. Indeed, suppose that $\sigma, \tilde{\sigma} \in A_n$. Suppose that σ and $\tilde{\sigma}$ are expressed as products of k and k' transpositions respectively. Then k and k' are even.

It follows that $\sigma\tilde{\sigma}$ can expressed as a product of $k + k'$ transpositions. Clearly, $k + k'$ is even. Hence $\sigma\tilde{\sigma}$ is even.

So A_n satisfies the closure axiom.

It remains to show that there are $\frac{n!}{2}$ even permutations and $\frac{n!}{2}$ odd permutations in S_n. To do this, we find a bijection from the subgroup A_n of even permutations to the set of odd permutations of degree n.