The composite $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$ of the permutations $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$ is defined to be the mapping resulting from performing the first mapping then the second one:

$$1 \mapsto 4 \mapsto 5, \qquad 2 \mapsto 3 \mapsto 1, \qquad 3 \mapsto 5 \mapsto 3, \qquad 4 \mapsto 1 \mapsto 2, \qquad 5 \mapsto 2 \mapsto 4,$$

i.e.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{pmatrix}.$$

The identity permutation is taken to be the identity mapping $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.

Proposition 5.4. *Let n be a positive integer. The permutations of degree n form a group S_n of order $n!$.*

Proof The closure axiom is satisfied since the composition

$$\sigma\tilde{\sigma} : \{1, 2, \dots, n\} \mapsto \{1, 2, \dots, n\}$$

of two permutations $\sigma, \tilde{\sigma} \in S_n$ (i.e. invertible mappings from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, n\}$) is indeed a permutation.

The associativity axiom is satisfied because the composition of mappings is associative.

The identity axiom is satisfied by the permutation $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$ of degree n.

For a permutation $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1\sigma & 2\sigma & 3\sigma & \cdots & n\sigma \end{pmatrix}$ of degree n, $\begin{pmatrix} 1\sigma & 2\sigma & 3\sigma & \cdots & n\sigma \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$ is a permutation of degree n which satisfies the inverse axiom.

Hence S_n is a group. By Example 1.6, it has $n!$ elements. □

Given any permutation σ of degree n given by

$$\begin{pmatrix} 1 & 2 & \dots & n \\ 1\sigma & 2\sigma & \dots & n\sigma \end{pmatrix},$$

we may look at the effect of repeatedly applying the permutation to any particular $i \in \{1, 2, \dots, n\}$:

$$i \mapsto i\sigma \mapsto i\sigma^2 \mapsto i\sigma^3 \mapsto \dots$$

Since the set $\{1, 2, \dots, n\}$ is finite, there must come a point when i is mapped back to itself. So there must exist $k \in \mathbb{N}$ with $i\sigma^k = i$.