For example, if $n = 3$ and $\sigma = (1\ 3\ 2)$, then

$$\begin{aligned} f_\sigma(x_1, x_2, x_3) &= (x_{1\sigma} - x_{2\sigma})(x_{1\sigma} - x_{3\sigma})(x_{2\sigma} - x_{3\sigma}) \\ &= (x_3 - x_1)(x_3 - x_2)(x_1 - x_2) \\ &= (x_1 - x_3)(x_2 - x_3)(x_1 - x_2) \\ &= f(x_1, x_2, x_3). \end{aligned}$$

Note that $f_\sigma(x_1, x_2, \ldots, x_n)$ contains the same number of bracketed terms as $f(x_1, x_2, \ldots, x_n)$. Each term in f_σ is either identical to one of the terms in f (if $i\sigma < j\sigma$) or to minus one of the terms in f (if $i\sigma > j\sigma$). Hence

$$f_\sigma(x_1, x_2, \ldots, x_n) = \pm f(x_1, x_2, \ldots, x_n).$$

We claim that

$$f_\sigma(x_1, x_2, \ldots, x_n) = (-1)^\sigma f(x_1, x_2, \ldots, x_n).$$

Indeed, suppose that σ is expressed as a product

$$\sigma = \tau_1 \tau_2 \ldots \tau_k$$

of transpositions $\tau_1, \tau_2, \ldots, \tau_k$. Each transposition τ_i negates f:

$$f_{\tau_i}(x_1, x_2, \ldots, x_n) = -f(x_1, x_2, \ldots, x_n) \quad \forall i = 1, 2, \ldots, k.$$

For example, if $n = 3$ and $\tau = (2\ 3)$, then

$$\begin{aligned} f_\tau(x_1, x_2, x_3) &= (x_1 - x_3)(x_1 - x_2)(x_3 - x_2) \\ &= -(x_1 - x_3)(x_1 - x_2)(x_2 - x_3) \\ &= -f(x_1, x_2, x_3). \end{aligned}$$

It follows that

$$\begin{aligned} f_\sigma(x_1, x_2, \ldots, x_n) &= \left(\cdots(f_{\tau_1})_{\tau_2}\cdots\right)_{\tau_k}(x_1, x_2, \ldots, x_n) \\ &= (-1)^k f(x_1, x_2, \ldots, x_n). \end{aligned}$$

In particular, it follows from the second equality that any expression of σ as a product of transpositions either always contains an even number of terms or always contains an odd number of terms.

Indeed, suppose that this is not the case, i.e. that σ can be expressed as a product of k transpositions and as a product of k' transpositions, with k even and k' odd. Then

$$f_\sigma(x_1, x_2, \ldots, x_n) = (-1)^k f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n),$$