Definition 5.23. A permutation σ is said to be *even* if it can be expressed as the product of an even number of transpositions. Otherwise σ is said to be *odd*. The *sign* of σ, denoted by $(-1)^\sigma$, is defined to be

$$(-1)^\sigma := \begin{cases} +1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Example 5.24. (1) Consider the permutation (5.2) of degree 6 in Example 5.16. Then

$$\sigma = (1\ 4\ 3)\,(2\ 5) = (1\ 4)\,(1\ 3)\,(2\ 5)\,.$$

Hence σ can be expressed as a product of an odd number of transpositions and thus is *odd*. So $(-1)^\sigma = -1$.

(2) The permutation (5.3) of degree 6 in Example 5.21 can be expressed as a product of an even number of transpositions (for example, $\sigma = (2\ 1)\,(2\ 6)\,(3\ 5)\,(3\ 4)$) and thus is *even*. So $(-1)^\sigma = +1$.

To check that the definition of odd and even permutations makes sense, we need to know that there are no permutations which can be expressed as products of both odd and even numbers of transpositions. The next proposition gives that this is indeed the case.

Proposition 5.25. *Let σ be a permutation of degree n. Any expression of σ as a product of transpositions either always contains an even number of terms or always contains an odd number of terms. Furthermore, if σ and $\tilde{\sigma}$ are permutations, then the sign of the composition $\sigma\tilde{\sigma}$ is the product of the signs of σ and $\tilde{\sigma}$:*

$$(-1)^{\sigma\tilde{\sigma}} = (-1)^\sigma\,(-1)^{\tilde{\sigma}}\,.$$

Proof Suppose that σ is a permutation of degree n. Let the polynomial $f(x_1, x_2, \ldots, x_n)$ in the variables $x_1, x_2, \ldots, x_n$ be defined by

$$f\,(x_1, x_2, \ldots, x_n) = \prod_{i<j} (x_i - x_j)\,.$$

For example, $f\,(x_1,\, x_2,\, x_3) = (x_1 - x_2)\,(x_1 - x_3)\,(x_2 - x_3)$.

Consider the polynomial

$$f_\sigma\,(x_1,\, x_2,\, \ldots,\, x_n) = \prod_{i<j} (x_{i\sigma} - x_{j\sigma})$$

obtained from applying σ to the indices $\{1,\, 2,\, \ldots,\, n\}$.