Continuing in this way, we see that

$$i_1 \mapsto i_2, \qquad i_2 \mapsto i_3, \qquad i_3 \mapsto i_4, \qquad \ldots \qquad i_{k-1} \mapsto i_k.$$

Finally, i_k is left unchanged by the first $k-2$ transpositions, then $i_k \mapsto i_1$ under the final transposition.

Hence $\sigma = \tilde{\sigma}$. □

Remark 5.20. It follows that each permutation can be expressed as a product of transpositions by first expressing it as a product of disjoint cycles and then expressing each cycle as a product of transpositions.

Example 5.21. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 3 & 4 & 2 \end{pmatrix} \tag{5.3}$$

of degree 6. To express it as a product of transpositions, we

- express it as a product of disjoint cycles:

$$\sigma = (1\ 6\ 2)\,(3\ 5\ 4)\,;$$

- express each cycle in turn as a product of transpositions:

$$\sigma = (1\ 6)\,(1\ 2)\,(3\ 5)\,(3\ 4)\,.$$

Note that using the equivalent expression

$$\sigma = (2\ 1\ 6)\,(3\ 5\ 4)$$

gives a different expression for σ as a product of transpositions:

$$\sigma = (2\ 1)\,(2\ 6)\,(3\ 5)\,(3\ 4)\,.$$

Remark 5.22. (1) As can be seen from the last example, a permutation can have more than one distinct representation as a product of transpositions.

(2) A permutation is expressed as a product of transpositions which are not disjoint. In general, these transpositions do not commute and must be written down in the correct order.

(3) In the last example, $\sigma = (2\ 1)\,(2\ 6)\,(3\ 5)\,(3\ 4)$. Hence

$$\sigma^{-1} = (3\ 4)^{-1}\,(3\ 5)^{-1}\,(2\ 6)^{-1}\,(2\ 1)^{-1} = (3\ 4)\,(3\ 5)\,(2\ 6)\,(2\ 1)\,.$$

In general, to obtain an expression for σ^{-1} as a product of transpositions, one can take such an expression for σ and reverse the order of the transpositions.