Example 5.16. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 3 & 2 & 6 \end{pmatrix} \tag{5.2}$$

of degree 6, which has disjoint cycle representation

$$\sigma = (1\ 4\ 3)\,(2\ 5)\,(6) = (1\ 4\ 3)\,(2\ 5) \quad \text{(the (6) is usually omitted).}$$

The order of σ is the lowest common multiple of 3 and 2, i.e. 6.

Definition 5.17. A *transposition of degree n, τ,* is a permutation of the form $(i\ j)$ for $i,\ j \in \{1,\ 2,\ \ldots,\ n\}$ with $i \neq j$.

Remarks 5.18. (1) A transposition of degree n is exactly a 2-cycle, i.e. it is a permutation of degree n which interchanges 2 elements of $\{1,\ 2,\ \ldots,\ n\}$ and leaves all other elements unaltered.

(2) Any transposition has order 2 and thus is its own inverse.

Proposition 5.19. *Any permutation σ may be expressed, in no way uniquely, as a product*

$$\sigma = \tau_1\,\tau_2\,\ldots\,\tau_k$$

of a finite number k of transpositions $\tau_1,\ \tau_2,\ \ldots,\ \tau_k$.

Proof Since, by Proposition 5.12, any permutation may be expressed as a product of disjoint cycles, it suffices to show that any cycle may be expressed (non necessarily uniquely) as a product of transpositions.

Indeed, suppose that σ is a cycle of length k:

$$\sigma = (i_1\ i_2\ i_3\ i_4\ \ldots\ i_k).$$

We take

$$\tilde{\sigma} = (i_1\ i_2)\,(i_1\ i_3)\,(i_1\ i_4)\ldots(i_1\ i_k)$$

and claim that $\sigma = \tilde{\sigma}$.

Indeed, consider $\tilde{\sigma}$. Then $i_1 \mapsto i_2$ under the first transposition, then i_2 is left unchanged because it does not appear in any of the remaining $k-2$ transpositions.

Further, $i_2 \mapsto i_1$ under the first transposition, then $i_1 \mapsto i_3$ under the second transposition, then i_3 is left unchanged by the remaining $k-3$ transpositions. So $i_2 \mapsto i_3$.

Similarly, i_3 is left unchanged by the first transposition, then $i_3 \mapsto i_1$ under the second transposition, then $i_1 \mapsto i_4$ under the third transposition, then i_4 is left unchanged by the remaining $k-4$ transpositions. So $i_3 \mapsto i_4$.