Example 5.13. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

of degree 5. Picking the element $1 \in \{1, 2, 3, 4, 5\}$ gives the cycle

$$\sigma = (1\ 4\ 5)$$

Picking the element $2 \in \{1, 2, 3, 4, 5\}$ which is not in the cycle $(1\ 4\ 5)$ gives the cycle

$$\sigma = (2\ 3).$$

Every element of $\{1, 2, 3, 4, 5\}$ is contained in either $(1\ 4\ 5)$ or $(2\ 3)$. Hence σ has disjoint cycle representation

$$\sigma = (1\ 4\ 5)\,(2\ 3)\,.$$

Remark 5.14. Disjoint cycles always commute. For example,

$$(1\ 4\ 5)\,(2\ 3) = (2\ 3)\,(1\ 4\ 5)\,.$$

Corollary 5.15. *The order of any permutation σ is the least common multiple of the lengths of the cycles in its expression as a product of disjoint cycles.*

Proof Let σ be a permutation and

$$\sigma = \gamma_1\,\gamma_2\,\ldots\,\gamma_k$$

be an expression of σ as a product of disjoint cycles. Since $\gamma_1, \gamma_2, \ldots, \gamma_k$ are disjoint, they commute. Hence for any positive integer m,

$$\sigma^m = \gamma_1^m\,\gamma_2^m\,\ldots\,\gamma_k^m.$$

Furthermore, the cycles $\gamma_1^m, \gamma_2^m, \ldots, \gamma_k^m$ are disjoint. Hence σ^m is the identity permutation if, and only if, each γ_i^m is an identity cycle. For $i \in \{1, 2, \ldots, k\}$, γ_i^m is the identity cycle if, and only if, m is a multiple of its order, i.e. length. Hence σ^m is the identity permutation if, and only if, m is a common multiple of the lengths of $\gamma_1, \gamma_2, \ldots, \gamma_k$. The order of σ is the least such m, i.e. the lowest common multiple of the lengths of $\gamma_1, \gamma_2, \ldots, \gamma_k$. $\square$