Remark 5.8. Given a cycle, we may easily compute its powers:

$$\sigma = (1\ 5\ 3\ 2) \Rightarrow \quad \sigma^2 = (1\ 3)\,(2\ 5)\,.$$

Similarly,

$$\sigma^3 = (1\ 2\ 3\ 5)\,, \qquad \sigma^4 = (1)\,(2)\,(3)\,(5)\,.$$

Hence σ has order 4.

Furthermore, it is easy to see that the order of any cycle is equal to its length.

Definition 5.9. A cycle of length k (i.e. one containing k elements $i_1, i_2, \ldots, i_k$) is called a *k-cycle*.

Remark 5.10. The inverse of any k-cycle is itself a k-cycle:

$$(i_1\ i_2\ \ldots\ i_{k-1}\ i_k)^{-1} = (i_k\ i_{k-1}\ \ldots\ i_2\ i_1)\,.$$

Example 5.11. The inverse of the 4-cycle $\sigma = (1\ 5\ 3\ 2)$ is the 4-cycle $\sigma^{-1} = (2\ 3\ 5\ 1) = (1\ 2\ 3\ 5)$.

Proposition 5.12. *Let n be a positive integer. Any permutation σ of degree n can be expressed as a product of disjoint cycles. For any two such expressions, the ordering of the disjoint cycles may be different, but the partitioning of the set $\{1, 2, \ldots, n\}$ amongst the cycles and the cyclic ordering of the pairs of cycles containing the same elements are the same.*

Proof Let n be a positive integer and σ be a permutation of degree n.

Pick any element $i \in \{1, 2, \ldots, n\}$, and let k be the least positive integer such that $i\sigma^k = i$. Note the cycle

$$\left(i\ i\sigma\ i\sigma^2\ \ldots\ i\sigma^{k-1}\right).$$

If $k < n$, we pick an element $i' \in \{1, 2, \ldots, n\}$ which is not in this cycle, let k' be the least positive integer such that $i'\sigma^{k'} = i'$, and note the cycle

$$\left(i'\ i'\sigma\ i'\sigma^2\ \ldots\ i'\sigma^{k'-1}\right),$$

which is disjoint from $\left(i\ i\sigma\ i\sigma^2\ \ldots\ i\sigma^{k-1}\right)$.

If $k + k' < n$, we pick element $i'' \in \{1, 2, \ldots, n\}$ which is not in either of the above cycles, let k'' be the least positive integer such that $i''\sigma^{k''} = i''$, and note the cycle

$$\left(i''\ i''\sigma\ i''\sigma^2\ \ldots\ i''\sigma^{k''-1}\right),$$

which is disjoint from $\left(i\ i\sigma\ i\sigma^2\ \ldots\ i\sigma^{k-1}\right)$ and $\left(i'\ i'\sigma\ i'\sigma^2\ \ldots\ i'\sigma^{k'-1}\right)$.

We continue until every element of $\{1, 2, \ldots, n\}$ is contained in one cycle. Then σ is the product of these mutually disjoint cycles since each element of $i \in \{1, 2, \ldots, n\}$ is affected by exactly one cycle. $\square$