Example 5.5. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix} \tag{5.1}$$

of degree 5. Its repeated action on each element of the set $\{1, 2, 3, 4, 5\}$ is:

$$\begin{aligned}
&1 \overset{\sigma}{\mapsto} 4 \overset{\sigma}{\mapsto} 5 \overset{\sigma}{\mapsto} 1,\\
&2 \overset{\sigma}{\mapsto} 3 \overset{\sigma}{\mapsto} 2,\\
&3 \overset{\sigma}{\mapsto} 2 \overset{\sigma}{\mapsto} 3,\\
&4 \overset{\sigma}{\mapsto} 5 \overset{\sigma}{\mapsto} 1 \overset{\sigma}{\mapsto} 4,\\
&5 \overset{\sigma}{\mapsto} 1 \overset{\sigma}{\mapsto} 4 \overset{\sigma}{\mapsto} 5.
\end{aligned}$$

It can be seen that the permutation σ can in fact be completely described by the following two cycles:

$$1 \to 4 \to 5 \to 1, \quad 2 \leftrightarrow 3.$$

These cycles represent the 'orbits' of the action of σ on the elements of the set $\{1, 2, 3, 4, 5\}$.

Definition 5.6. Let n be a positive integer. A *cycle* is a permutation of degree n which may be written in the form

$$(i_1\ i_2\ \ldots\ i_k),$$

where $i_1, i_2, \ldots, i_k$ are distinct elements of $\{1, 2, \ldots, n\}$ (i.e. $k \leq n$). By $(i_1\ i_2\ \ldots\ i_k)$, we mean the permutation which maps

$$i_1 \mapsto i_2, \qquad i_2 \mapsto i_3, \qquad \ldots \qquad i_{k-1} \mapsto i_k, \qquad i_k \mapsto i_1$$

and leaves every other element of $\{1, 2, \ldots, n\}$ unchanged.

Example 5.7. (1) Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}.$$

of degree 5. Then σ is the cycle (1 5 3 2). Note that this cycle may also be written as (5 3 2 1), (3 2 1 5) or (2 1 5 3).

(2) The permutation σ of degree 5 in (5.1) is (1 4 5) (2 3), or (2 3) (1 4 5), or (1 4 5) (3 2), etc.

(3) The cycle (1 2 3 5) is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}.$$

of degree 5.