Chapter 5

The Symmetric Group S_n

Definition 5.1. Let n be a positive integer. An invertible mapping $\sigma : \{1, 2, \ldots, n\} \mapsto \{1, 2, \ldots, n\}$ (such a mapping is invertible if, and only if, it is injective) is called a *permutation of degree n.*

Remark 5.2. We may define a mapping $\sigma : \{1, 2, \ldots, n\} \mapsto \{1, 2, \ldots, n\}$ by listing its effect on each element of the set $\{1, 2, \ldots, n\}$ in the form of an array

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ 1\sigma & 2\sigma & \ldots & n\sigma \end{pmatrix},$$

where for each $i \in \{1, 2, \ldots, n\}$, $i\sigma$ denotes the action of σ on i. Note that

$$\left.\begin{array}{c} \sigma \text{ is a permutation} \\ \text{of degree } n \end{array}\right\} \Leftrightarrow \sigma \text{ is invertible} \Leftrightarrow \left\{\begin{array}{l} \text{the elements in the bottom row are a} \\ \text{rearrangement of those in the top row.} \end{array}\right.$$

Example 5.3 (n = 5). Consider the mapping $\sigma : \{1, 2, 3, 4, 5\} \mapsto \{1, 2, 3, 4, 5\}$ given by $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$. This mapping is clearly invertible and is thus a permutation of degree 5. Furthermore,

$$1 \mapsto 4, \qquad 2 \mapsto 3, \qquad 3 \mapsto 5, \qquad 4 \mapsto 1, \qquad 5 \mapsto 2.$$

Hence the inverse of the permutation σ is given by the inverse mapping:

$$1 \mapsto 4, \qquad 2 \mapsto 5, \qquad 3 \mapsto 2, \qquad 4 \mapsto 1, \qquad 5 \mapsto 3,$$

and thus is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}.$$