To prove the second of claim, consider two cosets Ha and Hb which are not disjoint. We need to show that $Ha = Hb$.

Suppose that $c \in Ha \cap Hb$. Then $c \underset{H}{\sim} a$ and $c \underset{H}{\sim} b$. By symmetry, the first equivalence gives that $a \underset{H}{\sim} c$. Using the second equivalence and transitivity gives that $a \underset{H}{\sim} b$. Using symmetry gives that $b \underset{H}{\sim} a$.

If $d \in Ha$, then $d \underset{H}{\sim} a$. Since $a \underset{H}{\sim} b$, transitivity gives that $d \underset{H}{\sim} b$, i.e. that $d \in Hb$. So $Ha \subseteq Hb$.

If $d \in Hb$, then $d \underset{H}{\sim} b$. Since $b \underset{H}{\sim} a$, transitivity gives that $d \underset{H}{\sim} a$, i.e. that $d \in Ha$. So $Hb \subseteq Ha$.

Hence $Ha = Hb$. □

Remark 3.13. Proposition 3.12 gives a more efficient way of finding the right cosets of a subgroup H of a group G:

- write down the right coset $H1 = H$.
- while there exists an element of G which does not lie in any right coset that has been found so far, pick such an element $b \in G$ and write down the coset Hb (this contains b from the proof of Proposition 3.12).

Example 3.14. Consider the subgroup $4\mathbb{Z} = \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$ of the group $\mathbb{Z}$ of integers under addition. Using the algorithm in Remark 3.13 gives the right cosets of $4\mathbb{Z}$ in $\mathbb{Z}$:

$$\begin{aligned} 4\mathbb{Z} + 0 &= \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}, \\ 4\mathbb{Z} + 1 &= \{\ldots, -11, -7, -3, 1, 5, 9, 13, \ldots\}, \\ 4\mathbb{Z} + 2 &= \{\ldots, -10, -6, -2, 2, 6, 10, 14, \ldots\}, \\ 4\mathbb{Z} + 3 &= \{\ldots, -9, -5, -1, 3, 7, 11, 15, \ldots\}. \end{aligned}$$

The procedure is complete since every element of $\mathbb{Z}$ lies in one of the above four cosets. These cosets give the partition.

Lemma 3.15. *Let G be a group and H be a subgroup of G. For $a, b \in G$, $Ha = Hb$ if, and only if, $a \underset{H}{\sim} b$ (i.e. $ab^{-1} \in H$).*

Proof The $\Leftarrow$ implication was show in the proof of Proposition 3.12.

To show that the $\Rightarrow$ implication is true, pick $a, b \in G$ such that $Ha = Hb$.

From the proof of Proposition 3.12, $a \in Ha$. Hence $a \in Hb$, i.e. $a \underset{H}{\sim} b$. □

Lemma 3.16. *For any subgroup H of order $o(H)$ of a finite group G, each right coset of H contains $o(H)$ elements.*