It remains to show that the equivalence class of each $a \in G$ under the equivalence relation $\underset{H}{\sim}$ is the right coset Ha, i.e. that

$$\left\{b \in G : b \underset{H}{\sim} a\right\} = Ha \quad \forall a \in G. \tag{3.2}$$

Indeed, pick $a \in G$. We show firstly that $\left\{b \in G : b \underset{H}{\sim} a\right\} \subseteq Ha$ and then that $Ha \subseteq \left\{b \in G : b \underset{H}{\sim} a\right\}$.

To prove the first inclusion, pick $b \in G$ such that $b \underset{H}{\sim} a$ (such an element exists since $a \underset{H}{\sim} a$). Then $ba^{-1} \in H$. Hence $(ba^{-1})\,a \in Ha$.

Using the associativity, inverse and identity axioms of G gives that

$$\underbrace{(ba^{-1})\,a = b\,(a^{-1}a)}_{\text{by associativity axiom of } G} \quad \underbrace{= b1_G}_{\text{by inverse axiom of } G} \quad \underbrace{= b.}_{\text{by identity axiom of } G}$$

So $b \in Ha$, proving the first inclusion.

To prove the second inclusion, suppose that $b \in Ha$. Then there exists $h \in H$ such that $b = ha$. So $ba^{-1} = (ha)\,a^{-1}$. Using the associativity, inverse and identity axioms of G gives that

$$\underbrace{(ha)\,a^{-1} = h\,(aa^{-1})}_{\text{by associativity axiom of } G} \quad \underbrace{= h1_G}_{\text{by inverse axiom of } G} \quad \underbrace{= h.}_{\text{by identity axiom of } G}$$

So $ba^{-1} = h \in H$, i.e. $b \underset{H}{\sim} a$. □

We are now in a position to show the following.

Proposition 3.12. *The right cosets of a subgroup H of a group G provide a partition of G (i.e. they divide G into mutually disjoint subsets).*

Proof Let G be a group and H be a subgroup of G.

We need to show that each element of G is contained in (at least) one right coset of the subgroup H of G, and that any two cosets of the subgroup H of G are either equal or disjoint.

To prove the first of these claims, pick $a \in G$. Since $a \underset{H}{\sim} a$, the characterization (3.2) of the right coset Ha gives that

$$a \in \left\{b \in G : b \underset{H}{\sim} a\right\} = Ha.$$