(v) For any group G, the subset $\mathbf{1} = \{1_G\}$ consisting of the identity element of G is a subgroup of G and it is called the *trivial subgroup of* G.

(vi) For any group G, G itself is a subgroup of G (but *not* a proper subgroup).

Proposition 3.4. *A subset H of a* finite *group G is a subgroup of G if, and only if, H satisfies the closure and identity axioms with respect to the product operation of G.*

Proof The $\Rightarrow$ implication is true by the definition of a subgroup.

To show that the $\Leftarrow$ implication is true, suppose that G is a finite group of order n and $H = \{1_G, h_2, h_3, \ldots, h_m\}$ is a subset of G $(m \leq n)$ which operation of product of G. satisfies the closure and identity axioms with respect to the product operation of G.

We need to show that H satisfies the associativity and inverse axioms. closed under the operation of inverse of G. As in the proof of Proposition 3.2, H satisfies the associativity axiom. It remains to show that H satisfies the inverse axiom.

Pick $h \in H$. We need to show that there exists $h_i \in H$ such that $hh_i = 1_G = h_i h$.

Consider the subset $\{h1_G, hh_2, hh_3, \ldots, hh_m\}$ of H (that this is a subset of H follows from H being closed under the product operation of G).

By the Cancellation Law 1.9 in G, all elements of this subset are distinct. Since there are m of these elements, they are exactly the set of elements of H. Hence, since H satisfies the identity axiom, $1_G \in H = \{h1_G, hh_2, hh_3, \ldots, hh_m\}$. It follows that there exists $h_i \in H$ with $hh_i = 1_G$. Multiplying on the left by $h^{-1} \in G$ gives that $h^{-1}(hh_i) = h^{-1}1_G$.

Using the associativity, inverse and identity axioms of G gives that

$$h^{-1}(hh_i) \underset{\substack{\uparrow \\ \text{by associativity} \\ \text{axiom of } G}}{=} (h^{-1}h)\,h_i \underset{\substack{\uparrow \\ \text{by inverse} \\ \text{axiom of } G}}{=} 1_G h_i \underset{\substack{\uparrow \\ \text{by identity} \\ \text{axiom of } G}}{=} h_i.$$

Using the identity axiom of G gives that $h^{-1}1_G = h^{-1}$. Hence $h_i = h^{-1}$, and it follows by the identity axiom of G that $h_i h = h^{-1}h = 1_G$. ☐

Example 3.5. For the group $G = \{1, a, b, c, d, e\}$ with product defined by the product table

	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	b	1	e	c	d
b	b	1	a	d	e	c
c	c	d	e	1	a	b
d	d	e	c	b	1	a
e	e	c	d	a	b	1

(3.1)

$H = \{1, a, b\}$ is a subgroup of G since it satisfies the closure and identity axioms with respect to the product operation of G.