Chapter 3

Subgroups

Definition 3.1. The subset H of G is a *subgroup* of G if H is a group under the same product operation as G. If in addition $H \neq G$, then H is called a *proper subgroup* of G.

Proposition 3.2. *A a non-empty subset H of a group G is a subgroup of G if, and only if, H satisfies the closure and inverse axioms with respect to the product operation of G.*

Proof The $\Rightarrow$ implication is true by the definition of a subgroup.

To show that the $\Leftarrow$ implication is true, suppose that H is a non-empty subset of G which satisfies the closure and inverse axioms with respect to the product operation of G.

For any three elements $a, b, c \in H$, we have that $a, b, c \in G$. Hence, by the associativity axiom of G, we have that $a\,(bc) = (ab)\,c$. It follows that H satisfies the associativity axiom.

Pick an element $a \in H$. By the inverse axiom, $a^{-1} \in H$. Since $a, a^{-1} \in H$, axiom (i) give that $1_G = aa^{-1} \in H$. Hence H satisfies the identity exiom. □

Examples 3.3. (i) For each positive integer n, the group C_n is a subgroup of the group $\mathbb{C}^*$ of non-zero complex numbers under complex multiplication.

(ii) The groups $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ of integer, rational and real numbers respectively each under addition are subgroups of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ respectively (the latter being the group of complex numbers under addition).

(iii) For any integer n, the subset $n\mathbb{Z} = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\}$ of $\mathbb{Z}$ consisting of the integers which are divisible by n (i.e. $a \in n\mathbb{Z} \Leftrightarrow \exists b \in \mathbb{Z} : a = nb$) is a subgroup of the group $\mathbb{Z}$ under addition. For example, the subset $4\mathbb{Z} = \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\}$ of $\mathbb{Z}$ consisting of the integers which are divisible by 4 is a subgroup of the group $\mathbb{Z}$ under addition.

(iv) For each positive integer $n \geq 4$, the subset H of the group S_n consisting of the permutations σ of degree n under which the subset $\{2, 4\}$ of $\{1, 2, \ldots, n\}$ is invariant (i.e. $2 \mapsto 4$, $4 \mapsto 2$ or $2 \mapsto 2$, $4 \mapsto 4$; i.e. $2\sigma = 2$ or 4, $4\sigma = 2$ or 4) is a subgroup of S_n.