Chapter 2

Abelian groups

Definition 2.1. A group G is said to be *Abelian* if

$$ab = ba \quad \forall\, a,\, b \in G.$$

Examples 2.2. (1) For each positive integer n, the group C_n introduced in Example 1.4 *is* Abelian.

(2) For each positive integer $n \geq 3$ the group S_n introduced in Example 1.4 *is not* Abelian. Indeed, for example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

(3) The groups $\mathbb{R}^*$, $\mathbb{C}^*$ and $\mathbb{Q}^*$ introduced in Example 1.4 *are* Abelian.

(4) The groups $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ of integer, rational, real and complex numbers respectively each with operations of product, inverse and identity given by addition $(a+b)$, negation $(-a)$ and the number 0 respectively *are* Abelian.

(5) The groups $\mathrm{GL}\,(n,\mathbb{R})$, $\mathrm{GL}\,(n,\mathbb{Q})$ and $\mathrm{GL}\,(n,\mathbb{C})$ introduced in Example 1.4 *are not* Abelian for $n \geq 2$.

Remark 2.3. A finite group is Abelian if, and only if, its product table is symmetric about the leading diagonal.

Example 2.4. The set $G = \{a,\, b,\, c,\, d,\, e,\, f\}$ with product table (1.1) in Example 1.11 *is not* Abelian since $bc = e \neq f = cb$.

However, the group $G = \{1,\, a,\, b,\, c\}$ with product table

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

(2.1)