Example Consider an arbitrary permutation in S_4:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \square & \square & \square & \square \end{pmatrix}$$

$$\underset{\text{4 possibilities}}{\uparrow} \times \underset{\text{3 possibilities}}{\uparrow} \times \underset{\text{2 possibilities}}{\uparrow} \times \underset{\text{1 possibility}}{\uparrow} = 4! \text{ possibilities.}$$

(2) The groups $\mathbb{R}^*$, $\mathbb{C}^*$, $\mathbb{Q}^*$, $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{Q})$ and $\mathrm{GL}(n, \mathbb{C})$ (with $n \geq 2$) introduced in Example 1.4 are infinite.

One advantage of finite groups is that one may describe them by their *product (or Cayley) tables*. For example, for each positive integer n, the group of n^{th} roots of unity $C_n = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ introduced in Example 1.4 has product table

	1	ω	ω^2	ω^3	$\cdots$	ω^{n-3}	ω^{n-2}	ω^{n-1}
1	1	ω	ω^2	ω^3	$\cdots$	ω^{n-3}	ω^{n-2}	ω^{n-1}
ω	ω	ω^2	ω^3	ω^4	$\cdots$	ω^{n-2}	ω^{n-1}	1
ω^2	ω^2	ω^3	ω^4	ω^5	$\cdots$	ω^{n-1}	1	ω
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
ω^{n-3}	ω^{n-3}	ω^{n-2}	ω^{n-1}	1	$\cdots$	ω^{n-6}	ω^{n-5}	ω^{n-4}
ω^{n-2}	ω^{n-2}	ω^{n-1}	1	ω	$\cdots$	ω^{n-5}	ω^{n-4}	ω^{n-3}
ω^{n-1}	ω^{n-1}	1	ω	ω^2	$\cdots$	ω^{n-4}	ω^{n-3}	ω^{n-2}

By the following proposition, to know a group G we only need to know the set of elements in G and the product operation on G.

Proposition 1.7. *For any group G,*

(i) *the inverse $a^{-1} \in G$ of an element $a \in G$ is the unique element $b \in G$ for which $ab = 1 = ba$;*

(ii) *the identity element $1 \in G$ is the unique element of $e \in G$ for which*

$$ae = a = ea \quad \forall\, a \in G.$$

Proof (i) Fix $a \in G$ and suppose that $b \in G$ satisfies $ab = 1 = ba$.

We need to show that $b = a^{-1}$. Indeed, multiplying on the left by $a^{-1} \in G$ gives that $a^{-1}(ab) = a^{-1}1$.

Using the associativity, inverse and identity axioms gives that

$$a^{-1}(ab) \underset{\substack{\uparrow \\ \text{by associativity} \\ \text{axiom}}}{=} (a^{-1}a)\,b \underset{\substack{\uparrow \\ \text{by inverse} \\ \text{axiom}}}{=} 1b \underset{\substack{\uparrow \\ \text{by identity} \\ \text{axiom}}}{=} b.$$