Chapter 1

Group Axioms and Examples

Definition 1.1. A set G is a *group* if it satisfies the following four axioms:

$$\begin{array}{lll} \text{(i)} & \exists \text{ a binary operation } G \times G \mapsto G & \textbf{(closure)}, \\ & \qquad\qquad\qquad\quad (a,\, b) \mapsto ab & \\ \text{(ii)} & a\,(bc) = (ab)\,c \;\forall\, a,\, b,\, c \in G & \textbf{(associativity)}, \\ \text{(iii)} & \exists 1 \in G \text{ s.t. } a1 = a = 1a \;\forall\, a \in G & \textbf{(identity)}, \\ \text{(iv)} & \forall a \in G,\, \exists a^{-1} \in G \text{ s.t. } aa^{-1} = 1 = a^{-1}a & \textbf{(inverse)}. \end{array}$$

Remark 1.2. Written additively, the above four axioms become

$$\begin{array}{lll} \text{(i)} & \exists \text{ a binary operation } G \times G \mapsto G & \textbf{(closure)}, \\ & \qquad\qquad\qquad\quad (a,\, b) \mapsto a + b & \\ \text{(ii)} & a + (b + c) = (a + b) + c \;\forall\, a,\, b,\, c \in G & \textbf{(associativity)}, \\ \text{(iii)} & \exists 0 \in G \text{ s.t. } a + 0 = a = 0 + a \;\forall\, a \in G & \textbf{(identity)}, \\ \text{(iv)} & \forall a \in G,\, \exists - a \in G \text{ s.t. } a + (-a) = 0 = (-a) + a & \textbf{(inverse)}. \end{array}$$

Remarks 1.3. To define a group G, it is necessary to

- give the set of elements in G;
- find a binary operation which assigns to each pair $(a,\, b) \in G$ its **product** $ab \in G$;
- assign an element $a^{-1} \in G$, called the **inverse** of a, to each element in $a \in G$;
- select an element $1 \in G$, called the **identity** of G;
- verify that the associativity, inverse and identity axioms above hold.