8.1. Green's Theory for Regular $\mathcal{D}$-classes

If $e \in E(S)$ then H_e is a subgroup of S (by the Maximal Subgroup Theorem or Green's Theorem). If $e \,\mathcal{D}\, f$ then $|H_e| = |H_f|$ (by the Corollary to Green's Lemmas). We will show that $H_e \cong H_f$.

Lemma 8.8. *We have that*

(i) *If $a = axa$ then $ax, xa \in E(S)$ and $ax \,\mathcal{R}\, a \,\mathcal{L}\, xa$,*
(ii) *If $b \,\mathcal{R}\, f \in E(S)$, then b is regular;*
(iii) *If $b \,\mathcal{L}\, f \in E(S)$, then b is regular.*

Proof.

(i) We have already proven this.
(ii) If $b \,\mathcal{R}\, f$ then $fb = b$. Also, $f = bs$ for some $s \in S^1$. Therefore $b = fb = bsb$ and it follows that b is regular.
(iii) Dual to (ii).

□

From Lemma 8.8 an element $a \in S$ is regular if and only if it is $\mathcal{R}$-related to an idempotent. Dually, $a \in S$ is regular if and only if it is $\mathcal{L}$-related to an idempotent.

Lemma 8.9 (Regular $\mathcal{D}$-class Lemma)**.** *If $a \,\mathcal{D}\, b$ then if a is regular, so is b.*

Proof. Let a be regular with $a \,\mathcal{D}\, b$. Then $a \,\mathcal{R}\, c \,\mathcal{L}\, b$ for some $c \in S$.

a		e		c	
				f	
				b	

FIGURE 6. The egg box diagram of $\mathcal{D}$.

There exists $e = e^2$ with $e \,\mathcal{R}\, a \,\mathcal{R}\, c$ by (i) above. By (ii), c is regular. By (i), $c \,\mathcal{L}\, f = f^2$. By (iii), b is regular. □

Corollary 8.10. *[Corollary to Green's Lemmas] Let $e, f \in E(S)$ with $e \,\mathcal{D}\, f$. Then $H_e \cong H_f$.*

Proof. Suppose $e, f \in E(S)$ and $e \,\mathcal{D}\, f$. There exists $a \in S$ with $e \,\mathcal{R}\, a \,\mathcal{L}\, f$.
As $e \,\mathcal{R}\, a$ there exists $s \in S^1$ with $e = as$ and $ea = a$. So $a = asa$. Put $x = fse$. Then