

8.1. Green's Theory for Regular \mathcal{D} -classes

If $e \in E(S)$ then H_e is a subgroup of S (by the Maximal Subgroup Theorem or Green's Theorem). If $e \mathcal{D} f$ then $|H_e| = |H_f|$ (by the Corollary to Green's Lemmas). We will show that $H_e \cong H_f$.

Lemma 8.8. *We have that*

- (i) *If $a = axa$ then $ax, xa \in E(S)$ and $ax \mathcal{R} a \mathcal{L} xa$,*
- (ii) *If $b \mathcal{R} f \in E(S)$, then b is regular;*
- (iii) *If $b \mathcal{L} f \in E(S)$, then b is regular.*

Proof.

- (i) We have already proven this.
- (ii) If $b \mathcal{R} f$ then $fb = b$. Also, $f = bs$ for some $s \in S^1$. Therefore $b = fb = bsb$ and it follows that b is regular.
- (iii) Dual to (ii).

□

From Lemma 8.8 an element $a \in S$ is regular if and only if it is \mathcal{R} -related to an idempotent. Dually, $a \in S$ is regular if and only if it is \mathcal{L} -related to an idempotent.

Lemma 8.9 (Regular \mathcal{D} -class Lemma). *If $a \mathcal{D} b$ then if a is regular, so is b .*

Proof. Let a be regular with $a \mathcal{D} b$. Then $a \mathcal{R} c \mathcal{L} b$ for some $c \in S$.

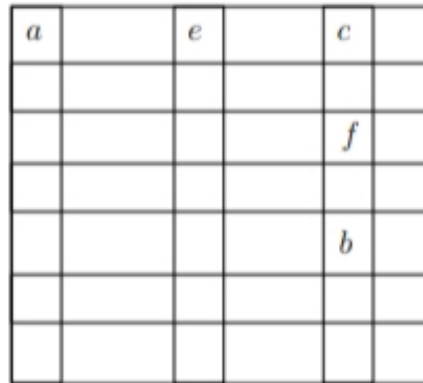


FIGURE 6. The egg box diagram of \mathcal{D} .

There exists $e = e^2$ with $e \mathcal{R} a \mathcal{R} c$ by (i) above. By (ii), c is regular. By (i), $c \mathcal{L} f = f^2$. By (iii), b is regular. □

Corollary 8.10. [Corollary to Green's Lemmas] *Let $e, f \in E(S)$ with $e \mathcal{D} f$. Then $H_e \cong H_f$.*

Proof. Suppose $e, f \in E(S)$ and $e \mathcal{D} f$. There exists $a \in S$ with $e \mathcal{R} a \mathcal{L} f$. As $e \mathcal{R} a$ there exists $s \in S^1$ with $e = as$ and $ea = a$. So $a = asa$. Put $x = fse$. Then