We want to show that fxe and ef are mutually inverse:

$$ef(fxe)ef = ef^2xe^2f = efxef = ef,$$
$$(fxe)ef(fxe) = fxe^2f^2xe = f(xefx)e = fxe.$$

Therefore we have $ef = (fxe)' = fxe \in E(S)$, so the product of any two idempotents is an idempotent. Therefore $E(S)$ is a band. Let $e, f \in E(S)$. Then

$$ef(fe)ef = ef^2e^2f = efef = ef$$

and $fe(ef)fe = fe$ similarly. Therefore we have $ef = (fe)' = fe$. □

Example 8.7.

(1) Let B be the Bicyclic Semigroup. Then
$$E(B) = \{(a, a) \mid a \in \mathbb{N}^0\},$$
and
$$(a, a)(b, b) = (t, t) = (b, b)(a, a)$$
where $t = \max\{a, b\}$. So $E(B)$ is commutative, and since B is regular, we have that it is inverse. Note that $(a, b)' = (b, a)$.

(2) $\mathcal{T}_X$ – we know $\mathcal{T}_X$ is regular. For $|X| \geqslant 2$ let $x, y \in X$ with $x \neq y$ we have $c_x, c_y \in E(\mathcal{T}_X)$. Then $c_xc_y \neq c_yc_x$ so $\mathcal{T}_X$ is not inverse.

(3) If S is a band, then S is regular. Furthermore we have
$$\begin{aligned} S \text{ is inverse} &\Leftrightarrow ef = fe \text{ for all } e, f \in E(S),\\ &\Leftrightarrow ef = fe \text{ for all } e, f \in S,\\ &\Leftrightarrow S \text{ is a semilattice.}\end{aligned}$$

(4) Let $\mathcal{M}^0 = \mathcal{M}^0(G; I, \Lambda; P)$. If $p_{\lambda i}, p_{\mu i}$ are both non-zero, then
$$(i, p_{\lambda i}^{-1}, \lambda), (i, p_{\mu i}^{-1}, \mu) \in E(\mathcal{M}^0)$$
and
$$(i, p_{\lambda i}^{-1}, \lambda)(i, p_{\mu i}^{-1}, \mu) = (i, p_{\mu i}^{-1}, \mu)(i, p_{\lambda i}^{-1}, \lambda)$$
if and only if $\lambda = \mu$. So for $\mathcal{M}^0$ to be inverse, for every $i \in I$ there must be *exactly* one $\lambda \in \Lambda$ with $p_{\lambda i} \neq 0$; dually for each $\kappa \in \Lambda$ there exists exactly one $j \in I$ with $p_{\kappa j} \neq 0$.

It is an *Exercise* to check that, conversely, if the above condition holds then $\mathcal{M}^0$ is inverse and isomorphic to a *Brandt* semigroup.